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A STUDY OF TWO-DIMENSIONAL PSEUDO-CHEBYSHEV WAVELETS AND THEIR APPLICATION TO FUNCTIONS OF HÖLDERS CLASS

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Abstract: In 2022, Shyam Lal, Susheel Kumar, and their collaborators introduced pseudo-Chebyshev wavelets in the context of one-dimension. Building on this foundation, the present study extends the framework to two dimensions. A two-dimensional pseudo-Chebyshev wavelet expansion is formulated and verified, and a novel algorithm is proposed for solving approximation problems. The method's effectiveness is demonstrated through illustrative examples and comparisons with standard Chebyshev wavelet methods. Error and convergence analyses are conducted for functions in the Hölder class, and the approximation error is estimated using generalized orthogonal projection operators. In this paper, we present several refinements of our current results, supported by illustrative examples that not only yield sharper bounds but also offer a more comprehensive and rigorous understanding of the underlying mathematical structure.

Keywords and Phrases: Pseudo-Chebyshev wavelets; Two-dimensional Pseudo-Chebyshev wavelets; Hölder class; Generalized orthogonal projection operator.

2020 Mathematics Subject Classification: 40A30, 42C15, 42A16, 65T60, 65L10.

1. Introduction

Wavelet theory has emerged as a fundamental tool in applied mathematics, offering efficient methods for analyzing localized features of functions at various scales. At the same time, Chebyshev polynomials introduced by Pafnuty Chebyshev in the 19th century have played a pivotal role in approximation theory, valued for their orthogonality and superior convergence characteristics.

To combine the advantages of both approaches, researchers have investigated hybrid wavelet constructions built upon classical orthogonal polynomials. In this context, the concept of pseudo-Chebyshev wavelets was first introduced by Shyam Lal, Susheel Kumar, and their collaborators in 2022 through their paper titled 'Error bounds of a function related to generalized Lipschitz class via the pseudo-Chebyshev wavelet and its applications in the approximation of functions [14].

This wavelet family was developed by modifying Chebyshev polynomials specifically by adjusting the collocation points and weights to form an effective wavelet system capable of accurately approximating both smooth and piecewise smooth functions. These wavelets exhibit strong convergence properties and offer rigorous error bounds for functions belonging to the generalized Lipschitz class [14].

Subsequent advancements employed these wavelets for approximating absolutely continuous signals and solving Abel's integral equations, highlighting their effectiveness within integral operator frameworks and many more [7, 8, 9, 10]. Their extension to two dimensions expanded their applicability to multivariate problems [5, 6], while the introduction of orthogonal projection operators based on extended pseudo-Chebyshev wavelet series offered refined theoretical insights and enhanced approximation estimates [4].

In the framework of the theory of special functions, the introduction of two-dimensional pseudo-Chebyshev wavelet-based constructions represents a meaningful step forward in the generalization and analytical deepening of this field. The development of such wavelet-driven operators demonstrates that the theory of special functions and polynomials can continue to evolve through the integration of modern approximation techniques and computational algorithms. In particular, this approach offers a rigorous pathway for extending classical families to multivariate contexts while preserving key structural properties such as orthogonality and convergence. Therefore, it is appropriate to strengthen the introduction by citing both foundational and recent contributions that have enriched the theory through methodologies such as operational calculus, q-analysis, and threshold-based frameworks, as well as recent advances in wavelet analysis, see ([3, 12, 13, 15, 16, 17, 19]).

Building on previous work, this study investigates the mathematical properties and applications of two-dimensional pseudo-Chebyshev wavelets through a generalized orthogonal projection operator based on Cesàro sums of order one. It focuses on error analysis, convergence behavior, and the practical implementation of this method for solving multivariable approximation problems.

2. Definitions and Preliminaries

2.1. Function of Hölders class in two variable

A two variable real valued function $f : \Omega^2 \to \mathbb{R}$, is said to be function of Hölder's class

$$i.e.\ f \in H^{(\alpha,\beta)}_{\Omega^2}(\mathbb{R}),$$

if there exists a number $\kappa > 0$ such that

$$|f(\omega+\nu,\varpi+\upsilon)-f(\omega,\varpi)| = \kappa(|\nu|^{\alpha}+|\upsilon|^{\beta}) = O(|\nu|^{\alpha}+|\upsilon|^{\beta}), [5, 6].$$

2.2. Two-Dimensional Pseudo-Chebyshev Wavelet (PCW)

The two dimensional PCW $\Psi_{(\eta,\vartheta;\eta',\vartheta')}$ are defined by

$$\begin{aligned}\Psi_{(\eta,\vartheta;\eta',\vartheta')}(\omega,\varpi) &:= \Psi^{(\varrho,\varrho')}_{(\eta,\vartheta;\eta',\vartheta')}(\omega,\varpi) = \psi^{\varrho}_{(\eta,\vartheta)}(\omega)\times\psi^{\varrho'}_{(\eta',\vartheta')}(\varpi)\\ &= \begin{cases} \frac{4}{\pi}2^{(\varrho+\varrho')/2}P_{(\vartheta+1/2)}(2^{\varrho}\omega-2\eta+1)P_{(\vartheta'+1/2)}(2^{\varrho}\varpi-2\eta'+1), \\ \qquad\qquad \text{for } \frac{\eta-1}{2^{\varrho-1}} \le \omega \le \frac{\eta}{2^{\varrho-1}},\ \&\ \frac{\eta'-1}{2^{\varrho'-1}} \le \varpi \le \frac{\eta'}{2^{\varrho'-1}}, \\ 0, \qquad\qquad \text{otherwise,}\end{cases}\end{aligned}$$

where ϑ, ϑ' are non negative integers and $\eta = 1, 2, 3, \cdots, 2^{\varrho-1}$, $\eta' = 1, 2, 3, \cdots, 2^{\varrho'-1}$ & ϱ, ϱ' are a positive integers.
$P_{(\vartheta+1/2)}(\omega) = \cos((\vartheta+1/2)(arc\cos\omega))$ $\vartheta = 0, 1, 2, \cdots$, and recurrence relations are given by,
$P_{\vartheta''+1/2}(\omega) = 2\omega P_{(\vartheta''-1/2)}(\omega) - P_{(\vartheta''-3/2)}(\omega)$, with $P_{\pm 1/2}(\omega) = \sqrt{\frac{1+\omega}{2}}$, $\vartheta'' \in \mathbb{N}, see[14]$.

2.3. Cesàro means

An infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable to the sum s by Cesàro means of an order one if,

$$\begin{aligned} t_n &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\alpha_{n,k}u_j \text{ where } \alpha_{n,k} = \begin{cases}\frac{1}{n+1} & \text{for } 0 \le j \le k \le n,\\ 0 & \text{for } j \ge k \ge n.\end{cases}\\ &= \sum_{j=0}^{k}\sum_{k=0}^{\infty}\alpha_{n,k}u_j = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{n+1}\sum_{j=0}^{k}u_j = \lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}s_k \text{ where } s_k = \sum_{j=0}^{k}u_j\\ &= s.\end{aligned}$$

This sum is denoted by symbolically $\sum\limits_{n=0}^{\infty} u_n = s(C,1)$, [2, 18].

Let $\sum\limits_{n=0}^{\infty}\sum\limits_{n'=0}^{\infty} u_{n,n'}$ be a double infinite series $(Bromwich[1], p.29)$, is said to be summable to the sum s by Cesàro means of an order one if,

$$t_{n,n'} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j'=0}^{\infty}\sum_{k'=0}^{\infty} \alpha_{(n,n';k,k')} u_{(j,j')}$$

$$\text{where}\alpha_{(n,n';k,k')} = \begin{cases} \frac{1}{(n+1)(n'+1)} & \text{for } \ 0 \le j \le k \le n, \ \& \ 0 \le j' \le k' \le n', \\ 0 & \text{for } j \ge k \ge n \ \& \ j' \ge k' \ge n'. \end{cases}$$

$$= \sum_{j=0}^{k}\sum_{k=0}^{\infty}\sum_{j'=0}^{k'}\sum_{k'=0}^{\infty} \alpha_{(n,n';k,k')} u_{(j,j')} = \lim_{(n,n')\to(\infty,\infty)} \sum_{k=0}^{n} \frac{1}{n+1} \sum_{j=0}^{k}\sum_{k'=0}^{n'} \frac{1}{n'+1} \sum_{j'=0}^{k'} u_{(j,j')}$$

$$= \lim_{(n,n')\to(\infty,\infty)} \frac{1}{(n+1)(n'+1)} \sum_{k=0}^{n}\sum_{k'=0}^{n'} s_{(k.k')} \text{ where } s_{(k,k')} = \sum_{j=0}^{k}\sum_{j'=0}^{k'} u_{(j,j')}$$

$$= \lim_{(n,n')\to(\infty,\infty)} \sum_{k=0}^{n}\sum_{k'=0}^{n'} \left(1 - \frac{1}{(n+1)}\right)\left(1 - \frac{1}{(n'+1)}\right) u_{(k.k')}$$

$$= s.$$

This sum is denoted by symbolically $\sum\limits_{n=0}^{\infty}\sum\limits_{n'=0}^{\infty} u_{(n,n')} = s(C,1,1)$.

Remark

(i) *If* $\sum\limits_{n=0}^{\infty} u_n = s,$ *then* $\sum\limits_{n=0}^{\infty} u_n = s(C,1).$

(ii) *An infinite series* $\sum\limits_{n=0}^{\infty} (-1)^n$ *is not convergent but* $\sum\limits_{n=0}^{\infty} (-1)^n = \frac{1}{2}(C,1),$ *see* [18].

(iii) *If* $\sum\limits_{n=0}^{\infty}\sum\limits_{n'=0}^{\infty} u_{n,n'} = s,$ *then* $\sum\limits_{n=0}^{\infty}\sum\limits_{n'=0}^{\infty} u_{n,n'} = s(C,1,1).$

(iv) *An infinite series* $\sum\limits_{n=0}^{\infty}\sum\limits_{n'=0}^{\infty} (-1)^{n+n'}$ *is not convergent but* $\sum\limits_{n=0}^{\infty}\sum\limits_{n'=0}^{\infty} (-1)^{n+n'} = \frac{1}{4}(C,1,1)$ *see* [11].

2.4. Generalized Orthogonal Projection Operator

An orthogonal projection operator is a surjective map $\Phi_\eta : L^2_\Omega \to V_\eta$ defined by (*see* [8])

$$\Phi_\eta(f) = \sum_{m=0}^{\infty} \left\langle f, \Psi_{(\eta,\vartheta)} \right\rangle_{w^\varrho_\eta} \Psi_{(\eta,\vartheta)}, \text{ fixed } \eta = 1,2,3,\cdots 2^{\varrho-1}, \ \varrho \in \mathbb{N}.$$

The two dimensional orthogonal projection operator $\Phi_{(\eta,\eta')} : L^2_{\Omega^2} \to V_{(\eta,\eta')}$ is given by

$$\begin{aligned}\Phi_{(\eta,\eta')}(f) &= \sum_{\vartheta'=0}^{\infty}\sum_{\vartheta=0}^{\infty} \alpha_{(\vartheta;\vartheta')}\Psi_{(\vartheta;\vartheta')},\\ &\text{fixed} \quad \eta = 1,2,3,\cdots 2^{\varrho-1} \quad \eta' = 1,2,3,\cdots 2^{\varrho'-1}, \quad \varrho,\varrho' \in \mathbb{N},\\ &= \sum_{\vartheta=0}^{\infty}\sum_{\vartheta'=0}^{\infty} \left\langle f, \Psi_{(\vartheta;\vartheta')} \right\rangle_{w^{(\varrho,\varrho')}_{(\eta;\eta')}} \Psi_{(\vartheta;\vartheta')},\\ &\text{where} \quad \alpha_{(\vartheta,\vartheta')} = \int_{\mathbb{R}}\int_{\mathbb{R}} f(\omega,\varpi)\Psi_{\vartheta;\vartheta'}(\omega,\varpi) w^{(\varrho,\varrho')}_{\eta;\eta'}(\omega,\varpi)d\omega d\varpi.\end{aligned}$$

The generalized orthogonal operator by $(C,1,1)$ method is denoted by $\Phi^G_{(\eta,\eta')}(f)$ and given by

$$\begin{aligned}\Phi^G_{(\eta,\eta')}(f) &:= \sum_{\vartheta'=0}^{n'}\sum_{\vartheta=0}^{n} \left(1-\frac{1}{n+1}\right)\left(1-\frac{1}{n'+1}\right)\alpha_{(\vartheta;\vartheta')}\Psi_{(\vartheta;\vartheta')},\\ &= \sum_{\vartheta'=0}^{n'}\sum_{\vartheta=0}^{n} \left(1-\frac{1}{n+1}\right)\left(1-\frac{1}{n'+1}\right)\left\langle f, \Psi_{(\vartheta;\vartheta')} \right\rangle_{w^{(\varrho,\varrho')}_{(\eta;\eta')}} \Psi_{(\vartheta;\vartheta')},\end{aligned}$$

2.5. Two-Dimensional Pseudo-Chebyshev Wavelet Series

A function $f \in L^2_{\Omega^2}(\mathbb{R})$ is expanded by two dimensional PCW series as [5, 6]:

$$\begin{aligned}f(\omega,\varpi) &= \sum_{\eta=1}^{\infty}\sum_{\vartheta=0}^{\infty}\sum_{\eta'=1}^{\infty}\sum_{\vartheta'=0}^{\infty} \alpha_{(\eta,\vartheta;\eta',\vartheta')}\psi^{(\omega)}_{(\eta,\vartheta)}\psi^{(\varpi)}_{(\eta',\vartheta')} \qquad (1)\\ &\text{where} \quad \alpha_{(\eta,\vartheta;\eta',\vartheta')} = \int\int f(\omega,\varpi)\psi^{(\omega)}_{(\eta,\vartheta)} w^\varrho_\eta(\omega)\psi^{(\varpi)}_{(\eta',\vartheta')} w^{\varrho'}_{\eta'}(\varpi)d\omega d\varpi.\end{aligned}$$

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2.6. Function Approximation

A two dimensional real valued function f defined on Ω^2 may be expanded in terms of the two dimensional PCW series (1). If an infinite series (1) is approximated by the generalized orthogonal projection operators $\Phi^G_{(\eta,\eta')}(f)$, then

$$\begin{aligned} f \approx f_0 &= \sum_{\eta=1}^{2^{\varrho-1}} \sum_{\vartheta=0}^{\wp-1} \sum_{\eta'=1}^{2^{\varrho'-1}} \sum_{\vartheta'=0}^{\wp'-1} \left(1-\frac{1}{n+1}\right)\left(1-\frac{1}{n'+1}\right) \left\langle f, \Psi_{(\vartheta;\vartheta')} \right\rangle_{w^{(\varrho,\varrho')}_{(\eta;\eta')}} \Psi_{(\vartheta;\vartheta')} \Psi_{\eta,\vartheta;\eta',\vartheta'} \\ &= \langle \Upsilon, \Psi \rangle = \Upsilon^{\tau}\Psi \text{ where } \Upsilon^{\tau} \text{ indicates transpose of a matrix } \Upsilon, \end{aligned}$$

where Υ and Ψ are $2^{\varrho-1}\wp 2^{\varrho'-1}\wp' \times 1$ matrices and $\langle \Upsilon, \Psi \rangle$ is an inner product of column vectors Υ and Ψ $(see[14])$.

2.7. Error of Wavelet Approximation

The error $\zeta^G_{(\aleph,\wp)}(f)$ of wavelet approximation of a function f by the generalized orthogonal projection operators $\Phi^G_{(\aleph,\wp)}(f)$ is defined by

$$\zeta_{(\aleph,\wp)}(f) = \inf_{\Phi^G_{(\aleph,\wp)}(f)} \|\Phi^G_{(\aleph,\wp)} - f\|_2 \quad \text{where } \aleph = 2^{\varrho-1} << \infty, \quad \text{and } \varrho \in \mathbb{N}.$$

If error $\zeta_{(\aleph,\wp)} \to 0$ as $\aleph \to \infty$ or $\wp \to \infty$ then $\Phi^G_{(\aleph,\wp)}(f)$ is called the best wavelet approximation of a function $f \in L^2_{\Omega}(\mathbb{R})$ $(see[3,20])$.

The error $\zeta_{(\aleph,\wp;\aleph',\wp')}$ of two dimensional PCW approximation of a function $f \in L^2_{\Omega^2}(\mathbb{R})$ by the operators $\Phi^G_{(\aleph,\wp;\aleph',\wp')}$ is given by

$$\zeta_{(\aleph,\wp;\aleph',\wp')} = \inf_{\Phi^{(G,f)}_{(\aleph,\wp;\aleph',\wp')}} \|f - \Phi^{(G,f)}_{(\aleph,\wp;\aleph',\wp')}\|_2.$$

If error $\zeta_{(\aleph,\wp;\aleph',\wp')} \to 0$ as $\aleph,\aleph' \to \infty$ or $\wp,\wp' \to \infty$ then $\Phi^G_{(\aleph,\wp;\aleph',\wp')}(f) = f_0$ is called the best wavelet approximation of a function $f \in L^2_{\Omega^2}(\mathbb{R})$.

3. Main results

3.1. In this section, two new theorems have been established in the following forms:

Theorem 1. *Let $f \in H^{(\alpha,\beta)}_{\Omega^2}(\mathbb{R})$ be a function, and let its two-dimensional Pseudo-Chebyshev wavelet series expansion be given by*

$$f \sim \sum_{\eta=1}^{\infty}\sum_{\vartheta=0}^{\infty}\sum_{\eta'=1}^{\infty}\sum_{\vartheta'=0}^{\infty} \alpha_{(\eta,\vartheta;\eta',\vartheta')}\Psi_{(\eta,\vartheta;\eta',\vartheta')},$$

where the coefficients are defined by

$$\alpha_{(\eta,\vartheta;\eta',\vartheta')} = \int_{\Omega^2} f\Psi_{(\eta,\vartheta;\eta',\vartheta')} w^{\varrho,\varrho'}_{\eta,\eta'} d\mu.$$

Then the error $\zeta_{(\aleph,\wp;\aleph',\wp')}$ *in the two-dimensional Pseudo-Chebyshev wavelet (PCW) approximation of the function* f*, using the generalized orthogonal projection operator* $\Phi^G_{(\eta,\eta')}(f)$*, is given by*

$$\zeta_{(\aleph,\wp;\aleph',\wp')} = O\left(\left(1-\frac{1}{\wp+1}\right)\left(1-\frac{1}{\wp'+1}\right)\left(\frac{1}{\aleph^{(\alpha+1)}}+\frac{1}{\aleph'^{(\beta+1)}}\right)\frac{1}{\sqrt{(\wp+1/2)(\wp'+1/2)}}\right)$$

Proof of Theorem 1. Since,

$$\begin{aligned}
(\zeta_{(\aleph,\wp;\aleph',\wp')})^2 &= \inf_{\Phi^{(G,f)}_{(\aleph,\wp;\aleph',\wp')}} \|f - \Phi^{(G,f)}_{(\aleph,\wp;\aleph',\wp')}\|_2^2, \\
&= \inf_{\Phi^{(G,f)}_{(\aleph,\wp;\aleph',\wp')}} \int_{\Omega^2} \left| f - \Phi^{(G,f)}_{(\aleph,\wp;\aleph',\wp')}\right|^2 d\mu, \\
&= \inf_{\{\wp,\wp'\}} \left(1-\frac{1}{\wp+1}\right)^2 \left(1-\frac{1}{\wp'+1}\right)^2, \\
&\times \left\| \sum_{n=1}^{\aleph}\sum_{m=\wp}^{\infty}\sum_{n'=1}^{\aleph'}\sum_{m'=\wp'}^{\infty} \varepsilon_{(n,m;n',m')}\Psi_{(n,m;n',m')} \right\|^2, \\
&= \inf_{\{\wp,\wp'\}} \left(1-\frac{1}{\wp+1}\right)^2 \left(1-\frac{1}{\wp'+1}\right)^2, \\
&\times \left\langle \sum_{n_1=1}^{\aleph}\sum_{n_1'=1}^{\aleph'}\sum_{m_1=\wp}^{\infty}\sum_{m_1'=\wp'}^{\infty} \varepsilon_{(n_1,m_1;n_1',m_1')}\Psi, \sum_{n_2=1}^{\aleph}\sum_{n_2'=1}^{\aleph'}\sum_{m_2=\wp}^{\infty}\sum_{m_2'=\wp'}^{\infty} \varepsilon_{(n_2,m_2;n_2',m_2')}\Psi \right\rangle, \\
&= \inf_{\{\wp,\wp'\}} \sum_{n_1=1}^{\aleph}\sum_{n_1'=1}^{\aleph'}\sum_{m_1=\wp}^{\infty}\sum_{m_1'=\wp'}^{\infty} \varepsilon_{(n_1,m_1;n_1',m_1')} \sum_{n_2=1}^{\aleph}\sum_{n_2'=1}^{\aleph'}\sum_{m_2=\wp}^{\infty}\sum_{m_2'=\wp'}^{\infty} \varepsilon_{(n_2,m_2;n_2',m_2')}, \\
&\times \left(1-\frac{1}{\wp+1}\right)^2 \left(1-\frac{1}{\wp'+1}\right)^2 \left\langle \Psi_{(n_1,m_1;n_1',m_1')}, \Psi_{(n_2,m_2;n_2',m_2')}\right\rangle_{w^{\varrho,\varrho'}_{\eta,\eta'}}
\end{aligned}$$

since $\left\langle \Psi_{(n_1,m_1;n_1',m_1')}, \Psi_{(n_2,m_2;n_2',m_2')}\right\rangle_{w^{\varrho,\varrho'}_{\eta,\eta'}} = \delta_{(n_1,n_2)}\delta_{(m_1,m_2)}\delta_{(n_1',n_2')}\delta_{(m_1',m_2')}$,

$$= \inf_{\{\wp,\wp'\}} \left(1-\frac{1}{\wp+1}\right)^2 \left(1-\frac{1}{\wp'+1}\right)^2 \sum_{n=1}^{\aleph}\sum_{n'=1}^{\aleph'}\sum_{m=\wp}^{\infty}\sum_{m'=\wp'}^{\infty} \left|\varepsilon_{(n,m;n',m')}\right|^2.$$

Now,

$$\begin{aligned}
\varepsilon_{(n,m;n',m')} &= \left\langle f, \Psi_{(n,m;n',m')} \right\rangle_{w_{\eta,\eta'}^{\varrho,\varrho'}} = \int_{\Omega^2} f \Psi_{(n,m;n',m')} w_{\eta,\eta'}^{\varrho,\varrho'} d\mu \\
&= \int_{\Omega}\int_{\Omega} f(x,y)\Psi_{(n,m;n',m')}(x,y) w_{\eta,\eta'}^{\varrho,\varrho'}(x,y)dxdy \\
&= \int_{\frac{\eta-1}{2^{\varrho-1}}}^{\frac{\eta}{2^{\varrho-1}}} \int_{\frac{\eta'-1}{2^{\varrho'-1}}}^{\frac{\eta'}{2^{\varrho'-1}}} f(x,y)\psi_{(\eta,\vartheta)}^{\varrho}(x) w_{\eta}^{\varrho}(x) \psi_{(\eta',\vartheta')}^{\varrho'}(y) w_{\eta'}^{\varrho'}(y)dxdy \\
\varepsilon_{(n,m;n',m')} &= \int_{\frac{\eta-1}{2^{\varrho-1}}}^{\frac{\eta}{2^{\varrho-1}}} \int_{\frac{\eta'-1}{2^{\varrho'-1}}}^{\frac{\eta'}{2^{\varrho'-1}}} \left(f(x,y) - f\left(\frac{2\eta-1}{2^{\varrho}}, \frac{2\eta'-1}{2^{\varrho'}}\right)\right) \\
&\times \psi_{(\eta,\vartheta)}^{\varrho}(x) w_{\eta}^{\varrho}(x) \psi_{(\eta',\vartheta')}^{\varrho'}(y) w_{\eta'}^{\varrho'}(y)dxdy \\
&+ \int_{\frac{\eta-1}{2^{\varrho-1}}}^{\frac{\eta}{2^{\varrho-1}}} \int_{\frac{\eta'-1}{2^{\varrho'-1}}}^{\frac{\eta'}{2^{\varrho'-1}}} f\left(\frac{2\eta-1}{2^{\varrho}}, \frac{2\eta'-1}{2^{\varrho'}}\right) \psi_{(\eta,\vartheta)}^{\varrho}(x) w_{\eta}^{\varrho}(x) \psi_{(\eta',\vartheta')}^{\varrho'}(y) w_{\eta'}^{\varrho'}(y)dxdy
\end{aligned}$$

Next, $f \in H_{\Omega^2}^{(\alpha,\beta)}\mathbb{R}$ and using Lemma 2.3 in [6], we have

$$\begin{aligned}
\left|\varepsilon_{(n,m;n',m')}\right| &\leq \left(\kappa\left(\frac{1}{2^{\varrho\alpha}} + \frac{1}{\varrho'\beta}\right) + \frac{4\kappa_0}{2^{\varrho}2^{\varrho'}}\right) \\
&\quad \times \int_{\frac{\eta-1}{2^{\varrho-1}}}^{\frac{\eta}{2^{\varrho-1}}} \int_{\frac{\eta'-1}{2^{\varrho'-1}}}^{\frac{\eta'}{2^{\varrho'-1}}} \psi_{(\eta,\vartheta)}^{\varrho}(x) w_{\eta}^{\varrho}(x) \psi_{(\eta',\vartheta')}^{\varrho'}(y) w_{\eta'}^{\varrho'}(y)dxdy \\
&\leq \frac{4}{\pi} max\left\{\kappa, 2\kappa_0\right\} \left(\frac{1}{\aleph^{(\alpha+1)}} + \frac{1}{\aleph'^{(\beta+1)}}\right)\left(\frac{1}{(m+1/2)(m'+1/2)}\right). \\
\left(\zeta_{(\aleph,\wp;\aleph',\wp')}\right)^2 &\leq \frac{16\kappa'^2}{\pi^2}\left(1-\frac{1}{\wp+1}\right)^2\left(1-\frac{1}{\wp'+1}\right)^2\left(\frac{1}{\aleph^{(\alpha+1)}} + \frac{1}{\aleph'^{(\beta+1)}}\right)^2 \\
&\times \sum_{m=\wp}^{\infty}\sum_{m'=\wp'}^{\infty}\left(\frac{1}{(m+1/2)(m'+1/2)}\right) \\
&= \frac{16\kappa'^2}{\pi^2}\left(1-\frac{1}{\wp+1}\right)^2\left(1-\frac{1}{\wp'+1}\right)^2\left(\frac{1}{\aleph^{(\alpha+1)}} + \frac{1}{\aleph'^{(\beta+1)}}\right)^2 \\
&\times \frac{1}{(\wp+1/2)(\wp'+1/2)} \text{ where,} \kappa' = \max\left\{\kappa, 2\kappa_0\right\} \quad \text{see Lemma 2.2 in [6].}
\end{aligned}$$

Therefore,

$$\zeta_{(\aleph,\wp;\aleph',\wp')} = O\left(\left(1-\frac{1}{\wp+1}\right)\left(1-\frac{1}{\wp'+1}\right)\left(\frac{1}{\aleph^{(\alpha+1)}} + \frac{1}{\aleph'^{(\beta+1)}}\right)\frac{1}{\sqrt{(\wp+1/2)(\wp'+1/2)}}\right)$$

Thus the establishment of Theorem 1 is now complete.

Theorem 2. *Let* $f \in H^{\alpha}_{\Omega}(\mathbb{R})$ *be a function, and let its Pseudo-Chebyshev wavelet series expansion be given by*

$$f \sim \sum_{\eta=1}^{\infty}\sum_{\vartheta=0}^{\infty} \alpha_{(\eta,\vartheta)}\Psi_{(\eta,\vartheta)},$$

where the coefficients are defined by

$$\alpha_{(\eta,\vartheta)} = \int_{\Omega} f\psi_{(\eta,\vartheta)} w^{\varrho}_{\eta} d\mu.$$

Then the error $\zeta_{(\aleph,\wp)}$ *in the two-dimensional Pseudo-Chebyshev wavelet (PCW) approximation of the function* f*, using the generalized orthogonal projection operator* $\Phi^{G}_{\eta}(f)$*, is given by*

$$\zeta_{(\aleph,\wp)} = O\left(\left(1-\frac{1}{\wp+1}\right)\frac{1}{\aleph^{(\alpha+1)}}\frac{1}{(\wp+1/2)}\right)$$

Proof of Theorem 2. The proof of Theorem 2 follows similarly to that of Theorem 1, taking into account the function class $f \in H^{\alpha}_{\Omega}(\mathbb{R})$ considered therein.

3.2. Corollaries

In this section, two new corollaries related to the Theorems 1 and 2, have been established in the following forms:

Corollary 1. *If* $f \in H^{(\alpha,\beta)}_{\Omega^2}(\mathbb{R})$ *be a function, and let its two-dimensional Pseudo-Chebyshev wavelet series expansion for* $\varrho = \varrho' = 1$*, be given by* $f \sim \sum_{\vartheta=0}^{\infty}\sum_{\vartheta'=0}^{\infty} \alpha_{(\vartheta,\vartheta')}\Psi_{(\vartheta,\vartheta')}$*, where the coefficients are defined by*

$$\alpha_{(\vartheta,\vartheta')} = \int_{\Omega^2} f\Psi_{(\vartheta,\vartheta')} w^{(1,1)}_{(1,1)} d\mu.$$

Then the error $\zeta^{(1)}_{(\wp,\wp')}$ *in the two-dimension, the Pseudo-Chebyshev wavelet (PCW) approximation of the function* f*, using the generalized orthogonal projection operator* $\Phi^{G}_{1}(f)$*, is given by*

$$\zeta^{(1)}_{(\wp,\wp')} = O\left(\left(1-\frac{1}{\wp+1}\right)\left(1-\frac{1}{\wp'+1}\right)\left(\frac{1}{2^{(\alpha+1)}}+\frac{1}{2^{(\beta+1)}}\right)\frac{1}{\sqrt{(\wp+1/2)(\wp'+1/2)}}\right)$$

Corollary 2. *If* $f \in H_{\Omega}^{\alpha}(\mathbb{R})$ *be a function, and let its Pseudo-Chebyshev wavelet series expansion in dimension for* $\varrho = 1$, *is given by* $f \sim \sum_{\vartheta=0}^{\infty} \alpha_{\vartheta}\psi_{\vartheta}$, *where the coefficients are defined by*

$$\alpha_{\vartheta} = \int_{\Omega} f\psi_{\vartheta} w_1^1 d\mu.$$

Then the error $\zeta_{\wp}^{(1)}$ *in the one-dimension, Pseudo-Chebyshev wavelet (PCW) approximation of the function* f, *using the generalized orthogonal projection operator* $\Phi_1^G(f)$, *is given by*

$$\zeta_{\wp}^{(1)} = O\left(\left(1 - \frac{1}{\wp+1}\right) \frac{1}{2^{(\alpha+1)}} \frac{1}{(\wp+1/2)}\right)$$

4. Illustrative Example

Example 1. In this example, we calculate the approximation of a function

$$f(t) = \begin{cases} 2t^{1/2} + 3t^{3/2} + 5t^{5/2} + 7t^{7/2}; \ t \in \Omega, \\ 0; \ t \notin \Omega. \end{cases}$$

Table 1: Comparison of PCW and CW approximations for different orders

t	0.00	0.25	0.50	0.75	1.00
f	0	1.5859375	3.9774756442	8.6737856848	17.0
PCW f_0^0	0	5.6015625	7.921805658	9.7021908518	11.203125
CW f_0^0	-1.9927209383	-0.99636046913	0	0.99636046913	1.9927209383
PCW f_0^1	0	0.9921875	4.6624853384	9.7021908518	15.8125
CW f_0^1	-0.84404941826	-1.5706962291	-1.14867152	0.42202470913	3.1413924583
PCW f_0^2	0	1.53125	3.90013584	8.7685072133	16.890625
CW f_0^2	-1.0712510935	-1.3434945539	-1.14867152	0.1948230339	3.3685941335
PCW f_0^3	0	1.5859375	3.9774756442	8.6737856848	17.0
CW f_0^3	-1.079900473	-1.3391698642	-1.1573208995	0.19914772366	3.359944754
PCW f_0^4	0	1.5859375	3.9774756442	8.6737856848	17.0
CW f_0^4	-1.0914030504	-1.3449211528	-1.1573208995	0.20489901236	3.3714473314

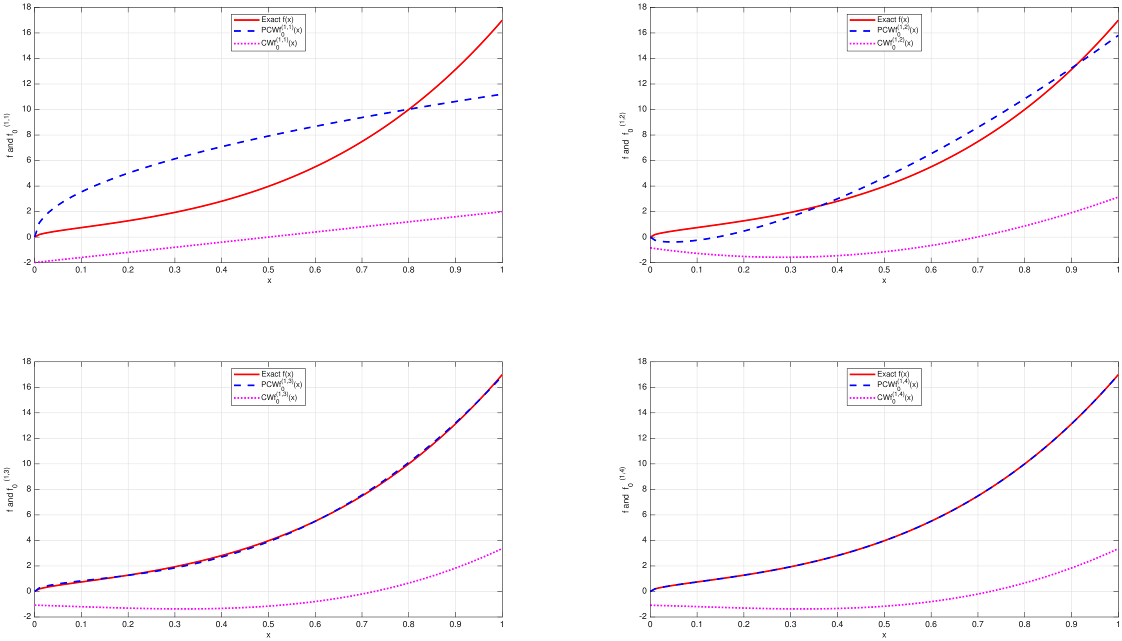

Figure 1: Graph of f and PCWf_0 and CWf_0 for different M

Table 2: Comparison of PCW and CW errors ζ^M for various M

t	0.00	0.25	0.50	0.75	1.00
PCWζ^1	0	4.015625	3.9443300138	1.028405167	5.796875
CWζ^1	1.9927209383	2.5822979691	3.9774756442	7.6774252156	15.007279062
PCWζ^2	0	0.59375	0.68500969427	1.028405167	1.1875
CWζ^2	0.84404941826	3.1566337291	5.1261471642	8.2517609757	13.858607542
PCWζ^3	0	0.0546875	0.077339804192	0.094721528539	0.109375
CWζ^3	1.0712510935	2.9294320539	5.1261471642	8.4789626509	13.631405866
PCWζ^4	0	0	0	0	0
CWζ^4	1.079900473	2.9251073642	5.1347965437	8.4746379611	13.640055246
PCWζ^5	0	0	0	0	0
CWζ^5	1.0914030504	2.9308586528	5.1347965437	8.4688866724	13.628552669

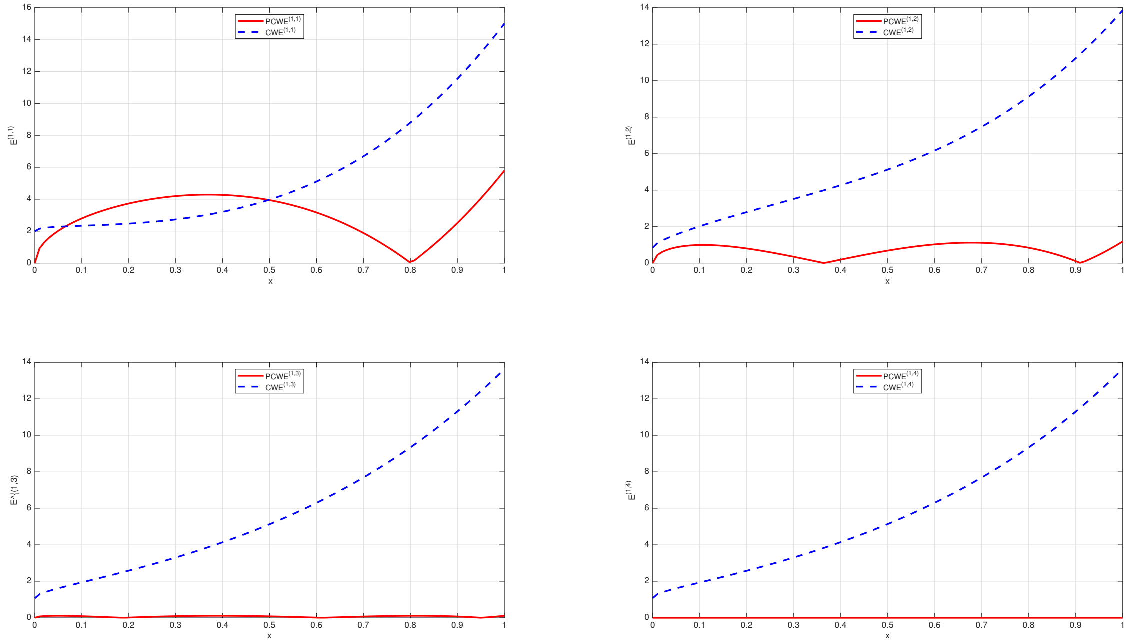

Figure 2: Graph of error functions ζ^M

Example 2. In this example, we consider a function defined over two variables, f.

$$f(t,u)=\begin{cases} 2t^{1/2}u^{3/2}+3t^{3/2}u^{1/2}+5t^{5/2}u^{7/2}+7t^{7/2}u^{5/2};\ (t,u)\in\Omega^2, \\ 0;\ t\notin\Omega^2. \end{cases}$$

Consider the Pseudo-Chebyshev series expansions of the function, for $\varrho=\varrho'\Rightarrow \eta=\eta'=1$

$$\begin{aligned} f &\sim \sum_{\vartheta=0}^{\infty}\sum_{\vartheta'=0}^{\infty}\alpha_{(\vartheta;\vartheta')}\psi_{\vartheta}\psi_{\vartheta'} \\ &= 6.1666\psi_0(t)\psi_0(u)+2.1376\psi_0(t)\psi_1(u)+0.4564\psi_0(t)\psi_2(u)+0.0383\psi_0(t)\psi_3(u) \\ &+ 2.3876\psi_1(t)\psi_0(u)+0.9664\psi_1(t)\psi_1(u)+0.2470\psi_1(t)\psi_2(u)+0.0192\psi_1(t)\psi_3(u) \\ &+ 0.5100\psi_2(t)\psi_0(u)+0.2684\psi_2(t)\psi_1(u)+0.0644\psi_2(t)\psi_2(u)+0.0038\psi_2(t)\psi_3(u) \\ &+ 0.0537\psi_3(t)\psi_0(u)+0.0268\psi_3(t)\psi_1(u)+0.0054\psi_3(t)\psi_2(u)+0\psi_3(t)\psi_3(u) \\ &= f_0, \end{aligned}$$

$$\text{where}\quad \alpha_{(\vartheta;\vartheta')}=\int_{\Omega^2}f\psi_{\vartheta}w_1^1\psi_{\vartheta'}w_1^1d\mu=\int_{t=0}^{1}\int_{u=0}^{1}f(t,u)\psi(t)w(2t-1)\psi(u)w(2u-1)dtdu,.$$

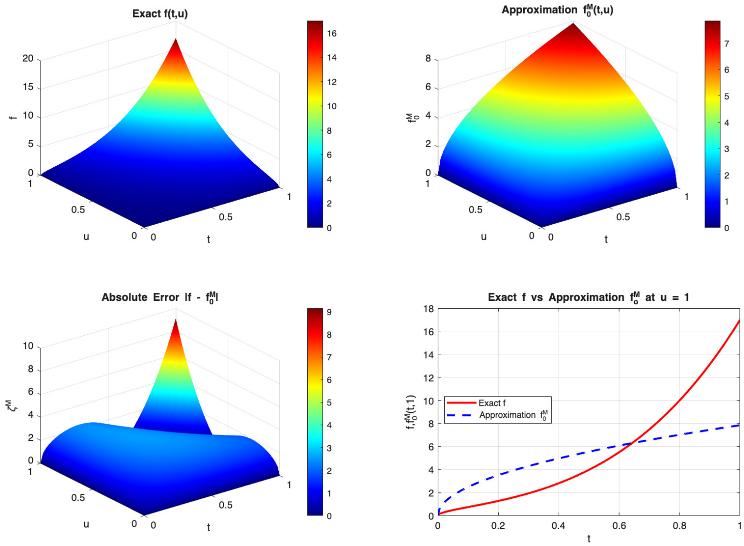

Figure 3: Graph of $f, f_0, |f - f_0|$ in 3D and f, f_0 in 2D for $M = 1$

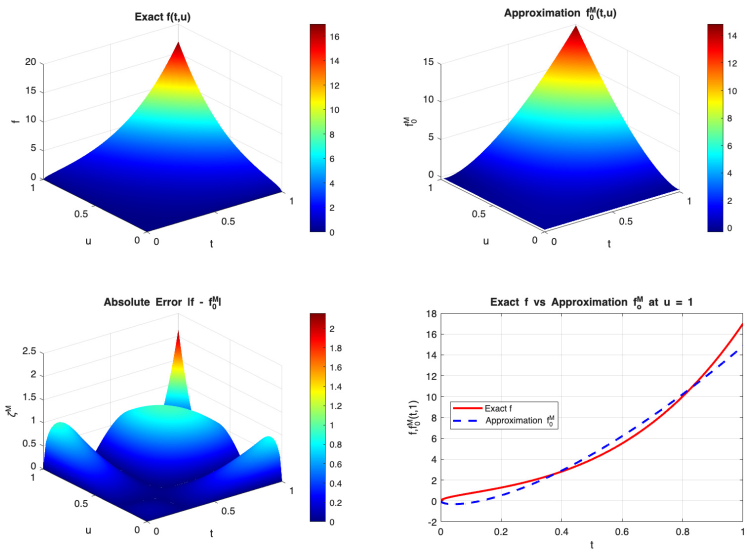

Figure 4: Graph of $f, f_0, |f - f_0|$ in 3D and f, f_0 in 2D for $M = 2$

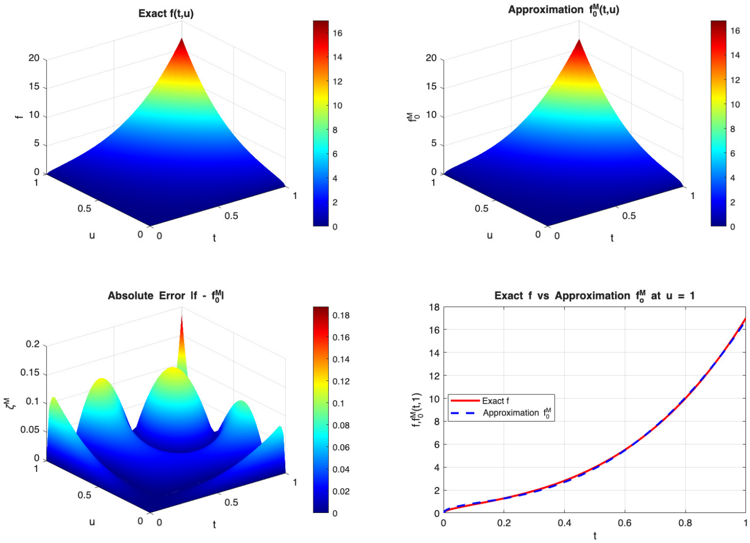

Figure 5: Graph of $f, f_0, |f - f_0|$ in 3D and f, f_0 in 2D for $M = 3$

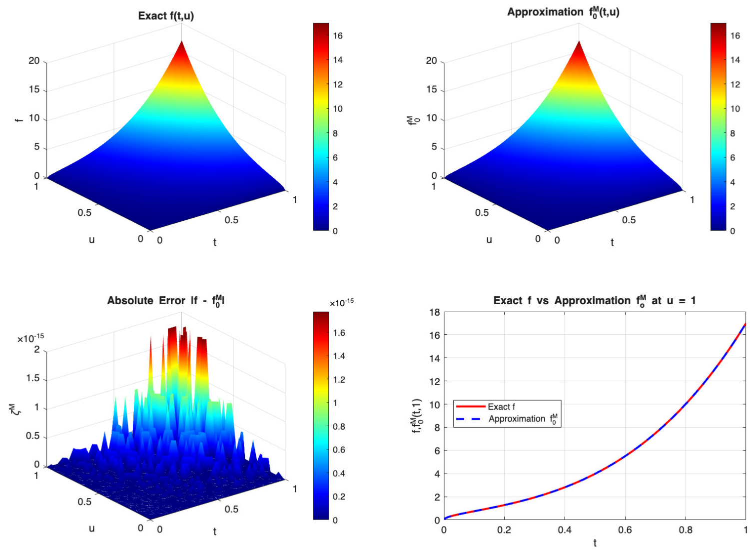

Figure 6: Graph of $f, f_0, |f - f_0|$ in 3D and f, f_0 in 2D for $M = 4$

5. Result Discussion and Conclusions

(i) Since, by Theorems 1 and 2, the errors satisfy

$$\zeta_{(\aleph,\wp;\aleph',\wp')} \to 0 \quad \text{as} \quad (\wp, \text{or } \wp', \text{or } \aleph, \text{or } \aleph') \to \infty,$$
$$\zeta_{(\aleph,\wp)} \to 0 \quad \text{as} \quad (\wp, \text{or } \aleph) \to \infty.$$

Therefore, the results presented in Theorems 1 and 2 demonstrate that the pseudo-Chebyshev wavelet approximations, based on the generalized orthogonal projection operators $\Phi_{(\eta,\eta')}$ and Φ_η respectively, provide highly accurate representations and are optimal within the framework of wavelet analysis [20]. Furthermore, the numerical results reported in Table 1 and Figures 1, 3, 4, 5, 6 along with the absolute errors displayed in Table 2 and Figure 2, strongly support the effectiveness of the proposed method.

(ii) In Example 2, by setting $u = 1$, the problem becomes analogous to that presented in Example 2. This correspondence is evident upon examining the solutions illustrated in Figures 1, 3, 4, 5, and 6.

(iii) Figures 1 and 2, together with Tables 1 and 2, clearly illustrate the comparative advantages of the proposed pseudo-Chebyshev wavelets over the classical Chebyshev wavelets in the context of the present examples. Specifically, the graphical representations in Figures 1 and 2 show that the pseudo-Chebyshev wavelets yield more accurate approximations to the target function, with smoother convergence behavior and reduced oscillations near the boundaries. Furthermore, the numerical results summarized in Tables 1 and 2 indicate lower approximation errors and improved stability in the pseudo-Chebyshev approach. These observations highlight the enhanced performance and suitability of the pseudo-Chebyshev wavelets for solving the considered class of problems.

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