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COMPUTATION THROUGH DEFINITE INTEGRATIONS FOR HYPERGEOMETRIC FUNCTIONS

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Abstract: In this article authors establish ten definite integrations for hypergeometric functions in association with Bessel function. Several closely-related results such as (for example) Generalized hypergeometric functions are also considered. These results provide some extensions in the scientific literature.

Keywords and Phrases: Bessel Function, Ramanujan's series.

2020 Mathematics Subject Classification: 33B50, 33C05, 33C10, 33D50, 33D60, 33D67.

1. Introduction

The Bessel function of first kind of order n is defined as:

$$J_n(\eta) = \frac{(\frac{\eta}{2})^n}{\Gamma(n+1)}\ {}_0F_1\Big(-; n+1; -\frac{\eta^2}{4}\Big) \tag{1.1}$$

We know following Ramanujan's series [1, 3, 4] are as:

$$R_4 \equiv \frac{4}{\pi} = 1 + \frac{7}{4}\Big(\frac{1}{2}\Big)^3 + \frac{13}{4^2}\Big(\frac{1.3}{2.4}\Big)^3 + \frac{19}{4^3}\Big(\frac{1.3.5}{2.4.6}\Big)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(6n+1)(\frac{1}{2})_n}{4^n (n!)^3} \tag{1.2}$$

$$R_5 \equiv \frac{16}{\pi} = 5 + \frac{47}{64}\Big(\frac{1}{2}\Big)^3 + \frac{89}{64^2}\Big(\frac{1.3}{2.4}\Big)^3 + \frac{131}{64^3}\Big(\frac{1.3.5}{2.4.6}\Big)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(42n+5)\left(\frac{1}{2}\right)_n^3}{64^n (n!)^3} \tag{1.3}$$

$$R_6 \equiv \frac{32}{\pi} = (5\sqrt{5}-1) + \Big(\frac{47\sqrt{5}+29}{64}\Big)\Big(\frac{1}{2}\Big)^3\Big(\frac{\sqrt{5}-1}{2}\Big)^8 +$$

$$+\Big(\frac{89\sqrt{5}+59}{64^2}\Big)\Big(\frac{1.3}{2.4}\Big)^3\Big(\frac{\sqrt{5}-1}{2}\Big)^{16} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(42\sqrt{5}\ n + 5\sqrt{5} + 30n - 1)\left(\frac{1}{2}\right)_n^3}{64^n (n!)^3}\Big(\frac{\sqrt{5}-1}{2}\Big)^{8n} \tag{1.4}$$

and

$$R_7 \equiv \frac{27}{4\pi} = 2 + 17\ \frac{1}{2}\ \frac{1}{3}\ \frac{2}{3}\Big(\frac{2}{27}\Big) + 32\ \frac{1.3}{2.4}\ \frac{1.4}{3.6}\ \frac{2.5}{3.6}\ \Big(\frac{2}{27}\Big)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(15n+2)\left(\frac{1}{2}\right)_n\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n}{(n!)^3}\Big(\frac{2}{27}\Big)^n \tag{1.5}$$

Qureshi *et al* derived the following formulae [2]:

$${}_2F_3\Big(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}, \frac{1}{2}, 1; -\frac{\pi^2}{4}\Big) = 0 \tag{1.6}$$

$${}_2F_3\Big(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}, \frac{1}{2}, 1; -\pi^2\Big) = -J_0(\pi) \tag{1.7}$$

$${}_2F_3\Big(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}, \frac{1}{2}, 1; -\frac{\pi^2}{9}\Big) = \frac{1}{2}\ J_0\Big(\frac{\pi}{3}\Big) \tag{1.8}$$

and

$${}_4F_3\Big(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{7}{6}; 1, 1, \frac{1}{6}; \frac{1}{4}\Big) = \frac{4}{\pi}. \tag{1.9}$$

2. Definite Integrals Associated with Hypergeometric Functions

In this section, we establish set of ten definite integrals associated with hypergeometric functions, as follows:

Theorem 2.1. *Each of the following assertion holds true:*

$$\int_0^\infty \frac{e^{-t}\ {}_1F_3\left(\frac{1}{4};\frac{1}{2},\frac{1}{2},1;-\frac{\pi^2 t}{4}\right)}{\sqrt[4]{t}}\ dt = 0 \tag{2.1}$$

$$\int_0^1 \frac{{}_1F_2\left(\frac{1}{4};\frac{1}{2},\frac{1}{2};-\frac{\pi^2 t}{4}\right)}{(1-t)^{\frac{3}{4}}\sqrt[4]{t}}\ dt = 0 \tag{2.2}$$

$$\int_{\iota\infty-\gamma}^{\iota\infty+\gamma} \frac{4^s\ \pi^{-2s}\ \Gamma(\frac{1}{4}-s)\ \Gamma(\frac{3}{4}-s)\ \Gamma(s)}{\Gamma(\frac{1}{2}-s)^2\ \Gamma(1-s)}\ ds = 0 \tag{2.3}$$

$$\int_0^\infty \frac{e^{-t}\ {}_1F_3\left(\frac{1}{4};\frac{1}{2},\frac{1}{2},1;-\pi^2 t\right)}{\sqrt[4]{t}}\ dt = -\Gamma\left(\frac{3}{4}\right)\ J_0(\pi) \tag{2.4}$$

$$\int_0^1 \frac{{}_1F_2\left(\frac{1}{4};\frac{1}{2},\frac{1}{2};-\pi^2 t\right)}{(1-t)^{\frac{3}{4}}\sqrt[4]{t}}\ dt = -\Gamma\left(\frac{3}{4}\right)\ \Gamma\left(\frac{1}{4}\right)\ J_0(\pi) \tag{2.5}$$

$$\int_{\iota\infty-\gamma}^{\iota\infty+\gamma} \frac{\pi^{-2s}\ \Gamma(\frac{1}{4}-s)\ \Gamma(\frac{3}{4}-s)\ \Gamma(s)}{\Gamma(\frac{1}{2}-s)^2\ \Gamma(1-s)}\ ds = -2\ \iota\ \Gamma\left(\frac{3}{4}\right)\ \Gamma\left(\frac{1}{4}\right)\ J_0(\pi) \tag{2.6}$$

$$\int_0^\infty \frac{e^{-t}\ {}_1F_3\left(\frac{1}{4};\frac{1}{2},\frac{1}{2},1;-\frac{\pi^2 t}{9}\right)}{\sqrt[4]{t}}\ dt = \frac{1}{2}\ \Gamma\left(\frac{3}{4}\right) J_0\left(\frac{\pi}{3}\right) \tag{2.7}$$

$$\int_0^1 \frac{{}_1F_2\left(\frac{1}{4};\frac{1}{2},\frac{1}{2};-\frac{\pi^2 t}{9}\right)}{(1-t)^{\frac{3}{4}}\sqrt[4]{t}}\ dt = \frac{1}{2}\ \Gamma\left(\frac{1}{4}\right)\ \Gamma\left(\frac{3}{4}\right) J_0\left(\frac{\pi}{3}\right) \tag{2.8}$$

$$\int_{\iota\infty-\gamma}^{\iota\infty+\gamma} \frac{9^s\ \pi^{-2s}\ \Gamma(\frac{1}{4}-s)\ \Gamma(\frac{3}{4}-s)\ \Gamma(s)}{\Gamma(\frac{1}{2}-s)^2\ \Gamma(1-s)}\ ds = \iota\ \Gamma\left(\frac{3}{4}\right)\ \Gamma\left(\frac{1}{4}\right)\ J_0\left(\frac{\pi}{3}\right) \tag{2.9}$$

and

$$\int_0^\infty e^{-t}\ \sqrt[6]{t}\ {}_3F_3\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1,\frac{1}{6};\frac{t}{4}\right)\ dt = \frac{2}{3\pi}\Gamma\left(\frac{1}{6}\right). \tag{2.10}$$

provided that each member of the assertions (2.1) to (2.10) exists.

Proofs. In this section, we provide proofs for the assertions (2.1) to (2.10), as given below:

We first prove our assertion (2.1) as:

$$\int_0^\infty \frac{e^{-t}\ {}_1F_3\Big(\frac{1}{4}; \frac{1}{2}, \frac{1}{2}, 1; -\frac{\pi^2 t}{4}\Big)}{\sqrt[4]{t}}\ dt = \Gamma\Big(\frac{3}{4}\Big) \sum_{k=0}^{\infty} \frac{(-\frac{1}{4})^k\ \pi^{2k}\ (\frac{1}{4})_k\ (\frac{3}{4})_k}{k!\ \Big((\frac{1}{2})_k\Big)^2\ (1)_k}$$
$$= \Gamma\Big(\frac{3}{4}\Big)\ {}_2F_3\Big(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}, \frac{1}{2}, 1; -\frac{\pi^2}{4}\Big)$$

Now using (1.6), we obtain:

$$\int_0^\infty \frac{e^{-t}\ {}_1F_3\Big(\frac{1}{4}; \frac{1}{2}, \frac{1}{2}, 1; -\frac{\pi^2 t}{4}\Big)}{\sqrt[4]{t}}\ dt = \Gamma\Big(\frac{3}{4}\Big) * 0 = 0$$

This completes our demonstration of the first assertion (2.1).
Next, we prove of second assertion (2.2), as:

$$\int_0^1 \frac{{}_1F_2\Big(\frac{1}{4}; \frac{1}{2}, \frac{1}{2}; -\frac{\pi^2 t}{4}\Big)}{(1-t)^{\frac{3}{4}}\sqrt[4]{t}}\ dt$$
$$= \Gamma\Big(\frac{1}{4}\Big)\ \Gamma\Big(\frac{3}{4}\Big) \sum_{k=0}^{\infty} \frac{{}_2F_3\Big(\frac{1}{4}+k, \frac{3}{4}+k; \frac{1}{2}+k, \frac{1}{2}+k, 1+k; z_0\Big)(\frac{1}{4})_k\ (\frac{3}{4})_k\ (-\frac{\pi^2}{4} - z_0)^k}{k!\Big((\frac{1}{2})_k\Big)^2\ (1)_k}$$
$$= \Gamma\Big(\frac{1}{4}\Big)\ \Gamma\Big(\frac{3}{4}\Big)\ {}_2F_3\Big(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}, \frac{1}{2}, 1; -\frac{\pi^2}{4}\Big)$$

Now applying (1.6), we obtain:

$$\int_0^1 \frac{{}_1F_2\Big(\frac{1}{4}; \frac{1}{2}, \frac{1}{2}; -\frac{\pi^2 t}{4}\Big)}{(1-t)^{\frac{3}{4}}\sqrt[4]{t}}\ dt = \Gamma\Big(\frac{1}{4}\Big)\ \Gamma\Big(\frac{3}{4}\Big) * 0 = 0.$$

This completes our demonstration of assertion (2.2).
Further, we prove assertion (2.3), as:

$$\int_{\iota\infty-\gamma}^{\iota\infty+\gamma} \frac{4^s\ \pi^{-2s}\ \Gamma(\frac{1}{4}-s)\ \Gamma(\frac{3}{4}-s)\ \Gamma(s)}{\Gamma(\frac{1}{2}-s)^2\ \Gamma(1-s)}\ ds$$

$$= 2\ \iota\ \Gamma\Big(\frac{1}{4}\Big)\ \Gamma\Big(\frac{3}{4}\Big) \frac{\Big(\prod_{k=1}^{3}\ \Gamma(b_k)\Big)\ \sum_{j=0}^{\infty} Res_{s=-j} \frac{4^s \pi^{-2s}\Gamma(s)\prod_{k=1}^{2}\Gamma(-s+a_k)}{\prod_{k=1}^{3}\Gamma(-s+b_k)}}{\prod_{k=1}^{2}\Gamma(a_k)}$$

$$= 2\ \iota\ \Gamma\Big(\frac{1}{4}\Big)\ \Gamma\Big(\frac{3}{4}\Big)\ {}_2F_3\Big(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}, \frac{1}{2}, 1; -\frac{\pi^2}{4}\Big)$$

Now using (1.6), we have

$$\int_{\iota\infty-\gamma}^{\iota\infty+\gamma} \frac{4^s\ \pi^{-2s}\ \Gamma(\frac{1}{4}-s)\ \Gamma(\frac{3}{4}-s)\ \Gamma(s)}{\Gamma(\frac{1}{2}-s)^2\ \Gamma(1-s)}\ ds = 2\ \iota\ \Gamma\Big(\frac{1}{4}\Big)\ \Gamma\Big(\frac{3}{4}\Big) * 0 = 0.$$

This completes our demonstration of assertion (2.3).
Further, we prove assertion (2.4), as:

$$\int_0^{\infty} \frac{e^{-t}\ {}_1F_3\Big(\frac{1}{4}; \frac{1}{2}, \frac{1}{2}, 1; -\pi^2 t\Big)}{\sqrt[4]{t}}\ dt = \Gamma\Big(\frac{3}{4}\Big) \sum_{k=0}^{\infty} \frac{(-1)^k\ \pi^{2k}\ (\frac{1}{4})_k\ (\frac{3}{4})_k}{k!\ \Big((\frac{1}{2})_k\Big)^2\ (1)_k}$$

$$= \Gamma\Big(\frac{3}{4}\Big)\ {}_2F_3\Big(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}, \frac{1}{2}, 1; -\pi^2\Big).$$

Now using (1.7), we have:

$$\int_0^{\infty} \frac{e^{-t}\ {}_1F_3\Big(\frac{1}{4}; \frac{1}{2}, \frac{1}{2}, 1; -\pi^2 t\Big)}{\sqrt[4]{t}}\ dt = \Gamma\Big(\frac{3}{4}\Big) * -J_0(\pi) = -\Gamma\Big(\frac{3}{4}\Big)\ J_0(\pi).$$

This completes our demonstration of assertion (2.4).
Further, we prove assertion (2.5), as:

$$\int_0^1 \frac{{}_1F_2\Big(\frac{1}{4}; \frac{1}{2}, \frac{1}{2}; -\pi^2 t\Big)}{(1-t)^{\frac{3}{4}}\sqrt[4]{t}}\ dt$$

$$= \Gamma\Big(\frac{1}{4}\Big)\ \Gamma\Big(\frac{3}{4}\Big) \sum_{k=0}^{\infty} \frac{{}_2F_3\Big(\frac{1}{4}+k, \frac{3}{4}+k; \frac{1}{2}+k, \frac{1}{2}+k, 1+k; z_0\Big)(\frac{1}{4})_k\ (\frac{3}{4})_k\ (-\pi^2-z_0)^k}{k!\Big((\frac{1}{2})_k\Big)^2\ (1)_k}$$

$$= \Gamma\Big(\frac{1}{4}\Big)\ \Gamma\Big(\frac{3}{4}\Big)\ {}_2F_3\Big(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}, \frac{1}{2}, 1; -\pi^2\Big)$$

Now applying (1.7), we obtain:

$$\int_0^1 \frac{{}_1F_2\Big(\frac{1}{4}; \frac{1}{2}, \frac{1}{2}; -\pi^2 t\Big)}{(1-t)^{\frac{3}{4}}\sqrt[4]{t}}\ dt = \Gamma\Big(\frac{1}{4}\Big)\ \Gamma\Big(\frac{3}{4}\Big) *\ -J_0(\pi) = -\Gamma\Big(\frac{1}{4}\Big)\ \Gamma\Big(\frac{3}{4}\Big) J_0(\pi).$$

This completes our demonstration of assertion (2.5).
Further, we prove assertion (2.6), as:

$$\int_{\iota\infty-\gamma}^{\iota\infty+\gamma} \frac{\pi^{-2s}\ \Gamma(\frac{1}{4}-s)\ \Gamma(\frac{3}{4}-s)\ \Gamma(s)}{\Gamma(\frac{1}{2}-s)^2\ \Gamma(1-s)}\ ds$$

$$= 2\ \iota\ \Gamma\Big(\frac{1}{4}\Big)\ \Gamma\Big(\frac{3}{4}\Big) \frac{\Big(\prod_{k=1}^{3}\ \Gamma(b_k)\Big)\ \sum_{j=0}^{\infty} Res_{s=-j} \frac{\pi^{-2s}\Gamma(s)\prod_{k=1}^{2}\Gamma(-s+a_k)}{\prod_{k=1}^{3}\Gamma(-s+b_k)}}{\prod_{k=1}^{2}\Gamma(a_k)}$$

$$= 2\ \iota\ \Gamma\Big(\frac{1}{4}\Big)\ \Gamma\Big(\frac{3}{4}\Big)\ {}_2F_3\Big(\frac{1}{4},\frac{3}{4};\frac{1}{2},\frac{1}{2},1;-\pi^2\Big)$$

Now using (1.7), we have:

$$\int_0^1 \frac{{}_1F_2\Big(\frac{1}{4};\frac{1}{2},\frac{1}{2};-\pi^2 t\Big)}{(1-t)^{\frac{3}{4}}\sqrt[4]{t}}\ dt = 2\ \iota\ \Gamma\Big(\frac{1}{4}\Big)\ \Gamma\Big(\frac{3}{4}\Big) * \big(-J_0(\pi)\big) = -2\ \iota\ \Gamma\Big(\frac{1}{4}\Big)\ \Gamma\Big(\frac{3}{4}\Big)\ J_0(\pi).$$

This completes our demonstration of assertion (2.6).
Further, we prove assertion (2.7), as:

$$\int_0^\infty \frac{e^{-t}\ {}_1F_3\Big(\frac{1}{4};\frac{1}{2},\frac{1}{2},1;-\frac{\pi^2 t}{9}\Big)}{\sqrt[4]{t}}\ dt = \Gamma\Big(\frac{3}{4}\Big)\sum_{k=0}^{\infty} \frac{(-\frac{1}{9})^k\ \pi^{2k}\ (\frac{1}{4})_k\ (\frac{3}{4})_k}{k!\ \Big((\frac{1}{2})_k\Big)^2\ (1)_k}$$

$$= \Gamma\Big(\frac{3}{4}\Big)\ {}_2F_3\Big(\frac{1}{4},\frac{3}{4};\frac{1}{2},\frac{1}{2},1;-\frac{\pi^2}{9}\Big).$$

Now using (1.8), we have:

$$\int_0^\infty \frac{e^{-t}\ {}_1F_3\Big(\frac{1}{4};\frac{1}{2},\frac{1}{2},1;-\frac{\pi^2 t}{9}\Big)}{\sqrt[4]{t}}\ dt = \Gamma\Big(\frac{3}{4}\Big) * \frac{1}{2}\ J_0\Big(\frac{\pi}{3}\Big) = \frac{1}{2}\ \Gamma\Big(\frac{3}{4}\Big) J_0\Big(\frac{\pi}{3}\Big).$$

This completes our demonstration of assertion (2.7).
Further, we prove assertion (2.8), as:

$$\int_0^1 \frac{{}_1F_2\Big(\frac{1}{4};\frac{1}{2},\frac{1}{2};-\frac{\pi^2 t}{9}\Big)}{(1-t)^{\frac{3}{4}}\sqrt[4]{t}}\ dt$$

$$= \Gamma\Big(\frac{1}{4}\Big)\ \Gamma\Big(\frac{3}{4}\Big)\sum_{k=0}^{\infty} \frac{{}_2F_3\Big(\frac{1}{4}+k,\frac{3}{4}+k;\frac{1}{2}+k,\frac{1}{2}+k,1+k;z_0\Big)(\frac{1}{4})_k\ (\frac{3}{4})_k\ (-\frac{\pi^2}{9}-z_0)^k}{k!\Big((\frac{1}{2})_k\Big)^2\ (1)_k}$$

$$= \Gamma\Big(\frac{1}{4}\Big)\ \Gamma\Big(\frac{3}{4}\Big)\ {}_2F_3\Big(\frac{1}{4},\frac{3}{4};\frac{1}{2},\frac{1}{2},1;-\frac{\pi^2}{9}\Big)$$

Now using (1.8), we got:

$$\int_0^1 \frac{{}_1F_2\Big(\frac{1}{4};\frac{1}{2},\frac{1}{2};-\frac{\pi^2 t}{9}\Big)}{(1-t)^{\frac{3}{4}}\sqrt[4]{t}}\ dt = \Gamma\Big(\frac{1}{4}\Big)\ \Gamma\Big(\frac{3}{4}\Big) * \frac{1}{2}\ J_0\Big(\frac{\pi}{3}\Big) = \frac{1}{2}\ \Gamma\Big(\frac{1}{4}\Big)\ \Gamma\Big(\frac{3}{4}\Big)\ J_0\Big(\frac{\pi}{3}\Big).$$

This completes our demonstration of assertion (2.8).
Further, we prove assertion (2.9), as:

$$\int_{\iota\infty-\gamma}^{\iota\infty+\gamma} \frac{9^s\ \pi^{-2s}\ \Gamma(\frac{1}{4}-s)\ \Gamma(\frac{3}{4}-s)\ \Gamma(s)}{\Gamma(\frac{1}{2}-s)^2\ \Gamma(1-s)}\ ds$$

$$= 2\ \Gamma\Big(\frac{1}{4}\Big)\ \Gamma\Big(\frac{3}{4}\Big) \frac{\Big(\prod_{k=1}^{3}\ \Gamma(b_k)\Big)\ \sum_{j=0}^{\infty} Res_{s=-j} \frac{9^s\pi^{-2s}\Gamma(s)\prod_{k=1}^{2}\Gamma(-s+a_k)}{\prod_{k=1}^{3}\Gamma(-s+b_k)}}{\prod_{k=1}^{2}\Gamma(a_k)}$$

$$= 2\ \Gamma\Big(\frac{1}{4}\Big)\ \Gamma\Big(\frac{3}{4}\Big)\ {}_2F_3\Big(\frac{1}{4},\frac{3}{4};\frac{1}{2},\frac{1}{2},1;-\frac{\pi^2}{9}\Big)$$

Now using (1.8), we obtain:

$$\int_{\iota\infty-\gamma}^{\iota\infty+\gamma} \frac{9^s\ \pi^{-2s}\ \Gamma(\frac{1}{4}-s)\ \Gamma(\frac{3}{4}-s)\ \Gamma(s)}{\Gamma(\frac{1}{2}-s)^2\ \Gamma(1-s)}\ ds$$

$$= 2\ \iota\ \Gamma\Big(\frac{1}{4}\Big)\ \Gamma\Big(\frac{3}{4}\Big) * \frac{1}{2}\ J_0\Big(\frac{\pi}{3}\Big) = \iota\ \Gamma\Big(\frac{3}{4}\Big)\ \Gamma\Big(\frac{1}{4}\Big)\ J_0\Big(\frac{\pi}{3}\Big).$$

This completes our demonstration of assertion (2.9).
Further, we prove assertion (2.10), as:

$$\int_0^\infty e^{-t}\ \sqrt[6]{t}\ {}_3F_3\Big(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1,\frac{1}{6};\frac{t}{4}\Big)\ dt$$

$$= -\Gamma\Big(\frac{7}{6}\Big) \frac{\Big(\prod_{k=1}^{3}\ \Gamma(b_k)\Big) \sum_{k=0}^{\infty} \frac{(\frac{1}{4}-t)^k\ G_{4,4}^{3,2}\left(\begin{array}{c|c} 1,\frac{1}{6}+k,1+k,1+k & \\ \frac{1}{2}+k,\frac{1}{2}+k,\frac{1}{2}+k,\frac{7}{6}+k & -\frac{1}{t}\end{array}\right)}{k!}}{\prod_{k=1}^{3}\ \Gamma(a_k)}$$

$$= \Gamma\Big(\frac{7}{6}\Big) \sum_{k=0}^{\infty} \frac{4^{-k}\ (\frac{7}{6})_k\ \Big((\frac{1}{2})_k\Big)^3}{k!\ (\frac{1}{6})_k\ \Big((1)_k\Big)^2}$$

$$= \Gamma\Big(\frac{7}{6}\Big)\ {}_4F_3\Big(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{7}{6};1,1,\frac{1}{6};\frac{1}{4}\Big)$$

Now using (1.9), we obtain

$$\int_0^\infty e^{-t}\ \sqrt[6]{t}\ {}_3F_3\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1,\frac{1}{6};\frac{t}{4}\right)\,dt=\Gamma\left(\frac{7}{6}\right)*\frac{4}{\pi}=\frac{1}{6}\,\Gamma\left(\frac{1}{6}\right)*\frac{4}{\pi}=\frac{2}{3\pi}\Gamma\left(\frac{1}{6}\right).$$

This completes our demonstration of assertion (2.10).
This obviously completes our proof of Theorem 2.1.

3. Conclusion

In our present investigation, we have made use of the constants, Gamma functions, and Bessel functions as well as the hypergeometric and the generalized hypergeometric functions with a view developing several definite integrals respectively.The results derived in this article are believed to be new and would extend and unify those that are available in the scientific literature.

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