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A STUDY OF INTEGRAL TRANSFORMS UNIFIED WITH WHITTAKER AND GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract: Integral transforms play a pivotal role in mathematical analysis and have become a central tool in the study of special functions. Motivated by their broad applicability, researchers have continually introduced new forms of these transforms. In this paper, we explore a novel integral transform involving the product of the Whittaker function $W_{k,m}(z)$ and a generalized hypergeometric function. The resulting integral is expressed in terms of the Srivastava triple hypergeometric function. Furthermore, several noteworthy special cases of the proposed integral transform are derived, highlighting its potential for broader applications in mathematical and applied contexts.

Keywords and Phrases: Whittaker function, Hypergeometric function, Srivastava Triple hypergeometric function and Laplace Transform.

2020 Mathematics Subject Classification: 35A22, 44A10, 33C15, 33C20, 33C70.

1. Introduction

In recent years, numerous integral transforms involving variety of special functions of mathematical physics have been established by many researchers (see for recent work, [1]–[4], [6]–[8], [10]–[13], [17]). Such transforms play an important role in many diverse fields of physics and engineering. Integral transforms involving Whittaker function and generalized hypergeometric function play crucial role in the problems of the various branches of physics and applied mathematics. Existing transforms involving Whittaker functions or hypergeometric functions usually remain limited to one-variable cases. The derived transformations in this article provide recurrence relations and reduction formulas connecting Whittaker-based integrals with classical functions. Due to the importance of such type of transforms, in this paper, we present (presumably new) certain potentially useful integral transforms involving Whittaker function and generalized hypergeometric function which are expressed in terms of sum of Srivastava triple hypergeometric function.

Integral transforms have been successfully used for almost two centuries for solving many problems in applied mathematics, mathematical physics and engineering science. The origin of the integral transforms including the Laplace and the Fourier transforms can be traced back to celebrate work of P.S. Laplace (1749-1827) on probability theory in the 1780s and to monumental treatise of J. Fourier (1768-1830) on La theorie Analytique de La Chaleur published in 1822. It may be relevant to point out that the Laplace transform is essentially a special case of Fourier transform for a class of functions defined on the positive real axis. One of the oldest and most commonly used integral transforms available in the mathematical literature is Laplace transform, which has effectively been used in finding the solution of the linear differential equations,integral equations, signal processing and control theory. For example Fourier and Laplace transform simplify complex signal and systems by converting the,m into different domains,making analysis easier for tasks such as filtering, image compression and system stability evaluations. In signal processing they enable frequency domain analysis and future instructions. In control system a transform differential equation into algebraic equations, facilitating the design and stability analysis of digital analogue controllers for application like robotics and aerospace. In mathematics, an integral transform T of the following form, where the input is a function f, and the output is the another function Tf. Mathematically

$$Tf(U) = \int_{t_1}^{t_2} K(t,u)f(t)dt. \tag{1.1}$$

The function of two variables K is called the kernel of the transform.

In particular, the Laplace transform is the integral transform with kernel

$$K(x,\varepsilon) = \chi(0,\infty)(x)e^{-z\varepsilon}, \tag{1.2}$$

because the Kernel is only non-zero for positive x_n, or the Laplace transform of a function $f(t)$ is the function

$$L[f](s) = \int_0^\infty (f(t))e^{-st}dt, \tag{1.3}$$

because the Kernel decays rapidly, the laplace transform makes sense for most functions.

Several quadratic transforms of Gauss hypergeometric series ${}_2F_1$ are found in literature. Some of these are [16; p. 111 (6), (17) and (4)]:

$${}_2F_1\left(\begin{matrix} a, & a+\frac{1}{2} & ; \\ & & & 2z-z^2 \\ \frac{1}{2}, & & ; \end{matrix}\right) = \left(1-\frac{1}{2}z\right)^{-2a} {}_2F_1\left(\begin{matrix} 2a. & 2a+\frac{1}{2} & ; \\ & & & \frac{3}{2-z} \\ \frac{1}{2} & & ; \end{matrix}\right), \tag{1.4}$$

$${}_2F_1\left(\begin{matrix} a, & a+\frac{1}{2} & ; \\ & & & z \\ c & & ; \end{matrix}\right) = \left(1+\sqrt{z}\right)^{-2a} {}_2F_1\left(\begin{matrix} 2a, & c-\frac{1}{2} & ; \\ & & & \frac{2\sqrt{z}}{1+\sqrt{z}} \\ 2c-\frac{1}{2} & & ; \end{matrix}\right), \tag{1.5}$$

$${}_2F_1\left(\begin{matrix} a, & b & ; \\ & & & z \\ 2b & & ; \end{matrix}\right) = \left(1-\frac{1}{2}z\right)^{-a} {}_2F_1\left(\begin{matrix} \frac{a}{2}, & \frac{a+1}{2} & ; \\ & & & \left(\frac{z}{2-z}\right)^2 \\ \frac{1}{2} & & ; \end{matrix}\right), \tag{1.6}$$

where we have used the standard hypergeometric series notation (see[14]):

$$\begin{aligned} {}_pF_q\left(\begin{matrix} a_1, & a_2,...,a_p; & \\ & & z \\ b_1, & b_2,...,b_q; & \end{matrix}\right) &= \sum_{n=0}^{\infty} \frac{(a_1)_n,...,(a_p)_n\ z^n}{(b_1)_n,...,(b_q)_n\ n!} \\ &= {}_pF_q[a_1,a_2,...,a_p;b_1,b_2,...,b_q;z], \end{aligned} \tag{1.7}$$

where $(a)_n = (a+1)(a+2)...(a+n-1)$ and $(a)_0 = 1$ is called Pochhammer's symbol [15].

A general triple hypergeometric series $F^{(3)}[x,y,z]$ defined by Srivastava [16; p. 69] is

$$F^{(3)}\left[\begin{matrix} (a) :: (h);(h');(h'') : (g);(g)';(g)'' & ; & \\ & & x,y,z \\ (b) :: (f);(f)';f'') : (e);(e)';(e)'' & ; & \end{matrix}\right]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}(h)_{m+n}(h')_{n+p}(h'')_{m+p}(g)_m(g')_n(g'')_p}{(b)_{m+n+p}(f)_{m+n}(f')_{n+p}(f'')_{m+p}(e)_m(e')_n(e'')_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}. \tag{1.8}$$

The Whitaker function of second kind $W_{k,\mu}(z)$ is defined as [18; p. 39 (24)] see also ([2], [6])

$$W_{k,\mu}(z) = z^{\mu+\frac{1}{2}} exp\left(-\frac{1}{2}z\right) \Psi\left(\mu - k + \frac{1}{2}, 2\mu+1; z\right). \tag{1.9}$$

Whitaker function has the following relations with other special functions

$$W_{\frac{1}{2}\alpha+\frac{1}{2}+n,\frac{1}{2}\alpha}(z) = (-1)^n\, n!\, e^{(-\frac{1}{2}z)} z^{\frac{1}{2}\alpha+\frac{1}{2}} L_n^{\alpha}(z), \tag{1.10}$$

where $L_n^{\alpha}(z)$ is a Laguerre Polynomial [15].

$$W_{\frac{1}{4}+n,\frac{1}{4}}(z^2) = (2)^{-n} e^{(-\frac{1}{2}z^2)} \sqrt{z} H_n(z), \tag{1.11}$$

where $H_n(z)$ is a Hermite polynomial [15].

$$W_{0,\nu}(2z) = \sqrt{\frac{2z}{\pi}} K_{\nu}(z), \tag{1.12}$$

where $K_{\nu}(z)$ is the modified Bessel function of second kind [15].

2. Main Transformation

In order to obtain, the main transformation, we establish an integral of the form

$$\int_0^{\infty} t^{\sigma-1} e^{-zt} W_{k,m}(\beta t)\, {}_pF_q\left(\begin{matrix} a_1, a_2, ..., a_p \;; \\ \\ b_1, b_2, ..., b_q \;; \end{matrix} \; x^2t^2\right) {}_uF_v\left(\begin{matrix} \lambda_1, \lambda_2, ..., \lambda_u \;; \\ \\ \mu_1, \mu_2, ..., \mu_v \;; \end{matrix} \; -y^2t^2\right) dt$$

$$= \frac{\beta^{m+\frac{1}{2}} \Gamma(A')}{(z+\frac{\beta}{2})^A} \left\{ \frac{\Gamma(A)}{\Gamma(C)} \right.$$

$$F^{(3)}\left[\begin{matrix} \frac{A}{2}, \frac{A+1}{2} :: & \frac{A'}{2}, \frac{A'+1}{2}; & -; & - & : a_1, a_2, ..., a_p; \lambda_1, \lambda_2, ..., \lambda_u; & \frac{B}{2}, & \frac{B+1}{2}; \\ \\ \frac{C}{2}, \frac{C+1}{2} :: & & & & : b_1, b_2, ..., b_q; \mu_1, \mu_2, ..., \mu_v; & \frac{1}{2} & ; \end{matrix}\right.$$

$$\left. \frac{16x^2}{(2z+\beta)^2}, \frac{-16y^2}{(2z+\beta)^2}, \frac{(2z-\beta)^2}{(2z+\beta)^2} \right]$$

$$+\left(\frac{2z-\beta}{2z+\beta}\right)\frac{B\Gamma(A+1)}{\Gamma(C+1)}$$

$$F^{(3)}\left[\begin{array}{l} \frac{A+1}{2},\frac{A+2}{2}::\ \frac{A'}{2},\frac{A'+1}{2};\ -;\ -\ :a_1,a_2,...,a_p;\lambda_1,\lambda_2,...,\lambda_u;\ \frac{B+1}{2},\ \frac{B+2}{2};\\ \frac{C+1}{2},\frac{C+2}{2}::\qquad :b_1,b_2,...,b_q;\mu_1,\mu_2,...,\mu_v;\ \frac{3}{2}\ ; \end{array}\right.$$

$$\left.\frac{16x^2}{(2z+\beta)^2},\frac{-16y^2}{(2z+\beta)^2},\frac{(2z-\beta)^2}{(2z+\beta)^2}\right]\Bigg\}, \tag{2.1}$$

where $A=m+\sigma+\frac{1}{2}, A'=\sigma-m+\frac{1}{2}, B=m-k+\frac{1}{2}, C=\sigma-k+1, Re(A)>0$, $Re(A')>0$ and $Re(z+\beta/2)>0$.

Proof. Denoting left hand side of (2.1) by I and expanding ${}_pF_q$ and ${}_uF_v$ in series and integrating term by term with the help of the result [3; p. 216 (16)] we get

$$I=\beta^{m+\frac{1}{2}}\sum_{r,s=0}^{\infty}\frac{(a_1)_r(a_2)_r...(a_p)_r(\lambda_1)_s(\lambda_2)_s...(\lambda_u)_s x^{2r}\ (-1)^s y^{2s}}{(b_1)_r(b_2)_r...(b_q)_r(\mu_1)_s(\mu_2)_s...(\mu_v)_s}$$

$$\frac{\Gamma(m+\sigma+2r+2s+\frac{1}{2})\Gamma(\sigma-m+2r+2s+\frac{1}{2})}{\Gamma(\sigma+2r+2s-k+1)(z+\frac{\beta}{2})^{m+\sigma+2r+2s+\frac{1}{2}}r!\ s!}$$

$$\times{}_2F_1\left(\begin{array}{ll} m+\sigma+2r+2s+\frac{1}{2},\ \ m-k+\frac{1}{2} & ; \\ \sigma+2r+2s-k+1 & ; \end{array}\ \frac{2z-\beta}{2z+\beta}\right). \tag{2.2}$$

Further expanding ${}_2F_1$ in series and making use of (1.8) , we arrive at the result (2.1).

3. Special Cases

(i). On taking $z=\frac{\beta}{2}$ in (2.1), we get

$$\int_0^\infty t^{\sigma-1}e^{-\frac{\beta}{2}t}W_{k,m}(\beta t){}_pF_q\left(\begin{array}{l} a_1,a_2,...,a_p\ ; \\ b_1,b_2,...,b_q\ ; \end{array}\ x^2t^2\right){}_uF_v\left(\begin{array}{l} \lambda_1,\lambda_2,...,\lambda_u\ ; \\ \mu_1,\mu_2,...,\mu_v\ ; \end{array}\ -y^2t^2\right)dt$$

$$=\frac{\beta^{m+\frac{1}{2}}\Gamma(A')}{\beta^A}\frac{\Gamma(A)}{\Gamma(C)}$$

$$F^{(2)}\left[\begin{array}{l} \frac{A}{2},\frac{A+1}{2},\frac{A'}{2},\frac{A'+1}{2}:\ a_1,a_2,...,a_p;\ \lambda_1,\lambda_2,...,\lambda_u; \\ \frac{C}{2},\frac{C+1}{2},-,-:\ \ b_1,b_2,...,b_q;\ \mu_1,\mu_2,...,\mu_v; \end{array}\ \frac{4x^2}{\beta^2},\frac{-4y^2}{\beta^2}\right], \tag{3.1}$$

where $F^{(2)}$ is the Kapmé de Fériet's function [15].

(ii). On taking $k = \frac{1}{2}\alpha + \frac{1}{2} + n$, $m = \frac{1}{2}\alpha$ in (2.1) and making use of (1.10), we get

$$\int_0^\infty t^{\sigma+\frac{1}{2}\alpha-\frac{1}{2}} e^{-(z+\frac{1}{2}\beta)t} L_n^\alpha(\beta t) {}_pF_q\left(\begin{matrix} a_1, a_2, ..., a_p; \\ \\ b_1, b_2, ..., b_q; \end{matrix} \quad x^2t^2\right) {}_uF_v\left(\begin{matrix} \lambda_1, \lambda_2, ..., \lambda_u; \\ \\ \mu_1, \mu_2, ..., \mu_v; \end{matrix} \quad -y^2t^2\right) dt$$

$$= \frac{\Gamma(A')}{(-1)^n n!(z+\frac{\beta}{2})^A} \left\{ \frac{\Gamma(A)}{\Gamma(C)} \right.$$

$$F^{(3)}\left[\begin{matrix} \frac{A}{2}, \frac{A+1}{2} :: \frac{A'}{2}, \frac{A'+1}{2}; -; - : a_1, a_2, ..., a_p; \lambda_1, \lambda_2, ..., \lambda_u; \frac{B}{2}, \frac{B+1}{2}; \\ \\ \frac{C}{2}, \frac{C+1}{2} :: \qquad : b_1, b_2, ..., b_q; \mu_1, \mu_2, ..., \mu_v; \frac{1}{2} ; \end{matrix}\right.$$

$$\left.\frac{16x^2}{(2z+\beta)^2}, \frac{-16y^2}{(2z+\beta)^2}, \frac{(2z-\beta)^2}{(2z+\beta)^2}\right] + \left(\frac{2z-\beta}{2z+\beta}\right) \times \frac{B\Gamma(A+1)}{\Gamma(C+1)}$$

$$\times F^{(3)}\left[\begin{matrix} \frac{A+1}{2}, \frac{A+2}{2} :: \frac{A'}{2}, \frac{A'+1}{2}; -; - : a_1, a_2, ..., a_p; \lambda_1, \lambda_2, ..., \lambda_u; \frac{B+1}{2}, \frac{B+2}{2}; \\ \\ \frac{C+1}{2}, \frac{C+2}{2} :: \qquad : b_1, b_2, ..., b_q; \mu_1, \mu_2, ..., \mu_v; \frac{3}{2} ; \end{matrix}\right.$$

$$\left.\left.\frac{16x^2}{(2z+\beta)^2}, \frac{-16y^2}{(2z+\beta)^2}, \frac{(2z-\beta)^2}{(2z+\beta)^2}\right]\right\}. \tag{3.2}$$

(iii). On taking $k = \frac{1}{4} + n$, $m = \frac{1}{4}$ in (2.1) and making use of (1.11), we get

$$\int_0^\infty t^{\sigma-\frac{3}{4}} e^{-(z+\frac{1}{2}\beta)t} H_n(\sqrt{\beta}t) {}_pF_q\left(\begin{matrix} a_1, a_2, ..., a_p; \\ \\ b_1, b_2, ..., , b_q; \end{matrix} \quad x^2t^2\right) {}_uF_v\left(\begin{matrix} \lambda_1, \lambda_2, ..., \lambda_u; \\ \\ \mu_1, \mu_2, ..., \mu_v; \end{matrix} \quad -y^2t^2\right) dt$$

$$= \frac{2^n \beta^{\frac{1}{2}} \Gamma(A')}{(z+\frac{\beta}{2})^A} \left\{ \frac{\Gamma(A)}{\Gamma(C)} \right.$$

$$F^{(3)}\left[\begin{matrix} \frac{A}{2}, \frac{A+1}{2} :: \frac{A'}{2}, \frac{A'+1}{2}; -; - : a_1, a_2, ..., a_p; \lambda_1, \lambda_2, ..., \lambda_u; \frac{B}{2}, \frac{B+1}{2}; \\ \\ \frac{C}{2}, \frac{C+1}{2} :: \qquad : b_1, b_2, ..., b_q; \mu_1, \mu_2, ..., \mu_v; \frac{1}{2} ; \end{matrix} \quad \frac{16x^2}{(2z+\beta)^2},\right.$$

$$\left.\frac{-16y^2}{(2z+\beta)^2}, \frac{(2z-\beta)^2}{(2z+\beta)^2}\right] + \left(\frac{2z-\beta}{2z+\beta}\right) \frac{B\Gamma(A+1)}{\Gamma(C+1)}$$

$$F^{(3)}\left[\begin{array}{l} \frac{A+1}{2},\frac{A+2}{2}::\ \frac{A'}{2},\frac{A'+1}{2};\ -;\ -\ : a_1,a_2,...,a_p;\lambda_1,\lambda_2,...,\lambda_u;\ \frac{B+1}{2},\ \frac{B+2}{2}; \\ \frac{C+1}{2},\frac{C+2}{2}::\ \qquad\qquad : b_1,b_2,...,b_q;\mu_1,\mu_2,....\mu_v;\ \frac{3}{2}\ ; \end{array}\right.$$

$$\left.\left.\frac{16x^2}{(2z+\beta)^2},\frac{-16y^2}{(2z+\beta)^2},\frac{(2z-\beta)^2}{(2z+\beta)^2}\right]\right\}. \tag{3.3}$$

(iv). On taking $k=0, m=\nu$ in (2.1) and making use of (1.12), we get

$$\int_0^\infty t^{\sigma-\frac{1}{2}}e^{-zt}K_\nu\left(\frac{\beta t}{2}\right){}_pF_q\left(\begin{array}{ll} a_1,a_2,...,a_p & ; \\ & x^2t^2 \\ b_1,b_2,...,b_q & ; \end{array}\right){}_uF_v\left(\begin{array}{ll} \lambda_1,\lambda_2,...,\lambda_u & ; \\ & -y^2t^2 \\ \mu_1,\mu_2,...,\mu_v & ; \end{array}\right)dt$$

$$=\frac{\sqrt{\pi}\beta^\nu\Gamma(A')}{(z+\frac{1}{2}\beta)^A}\left\{\frac{\Gamma(A)}{\Gamma(C)}\right.$$

$$F^{(3)}\left[\begin{array}{l} \frac{A}{2},\frac{A+1}{2}::\ \frac{A'}{2},\frac{A'+1}{2};-;-\ : a_1,a_2,...,a_p;\lambda_1,\lambda_2,...,\lambda_u;\ \frac{B}{2},\ \frac{B+1}{2}; \\ \frac{C}{2},\frac{C+1}{2}::\ \qquad\qquad : b_1,b_2,...,b_q;\mu_1,\mu_2,...,\mu_v;\ \frac{1}{2}\ ; \end{array}\right.\frac{16x^2}{(2z+\beta)^2},$$

$$\left.\frac{-16y^2}{(2z+\beta)^2},\frac{(2z-\beta)^2}{(2z+\beta)^2}\right]+\left(\frac{2z-\beta}{2z+\beta}\right)\frac{B\Gamma(A+1)}{\Gamma(C+1)}$$

$$F^{(3)}\left[\begin{array}{l} \frac{A+1}{2},\frac{A+2}{2}::\ \frac{A'}{2},\frac{A'+1}{2};\ -;\ -\ : a_1,a_2,...,a_p;\lambda_1,\lambda_2,...,\lambda_u;\ \frac{B+1}{2},\ \frac{B+2}{2}; \\ \frac{C+1}{2},\frac{C+2}{2}::\ \qquad\qquad : b_1,b_2,...,b_v;\mu_1,\mu_2,...,\mu_q;\ \frac{3}{2}\ ; \end{array}\right.$$

$$\left.\left.\frac{16x^2}{(2z+\beta)^2},\frac{-16y^2}{(2z+\beta)^2},\frac{(2z-\beta)^2}{(2z+\beta)^2}\right]\right\}. \tag{3.4}$$

(v). On taking $p=u=1, q=v=2, a_1=1, b_1=\frac{3}{2}, b_2=\delta, \lambda_1=1, \mu_1=\frac{3}{2}, \mu_2=\mu,$ in (2.1), using the following results [14; p.595 (11)] and [14; p. 608 (13)]

$${}_1F_2(a;b_1,b_2;z)=\frac{\sqrt{\pi}}{2}\Gamma(b)z^{1/2-b}L_{b-3/2}(2z)$$

$${}_1F_2(a;b_1,b_2;-z^2)=\frac{\sqrt{\pi}}{2}\Gamma(b)z^{1/2-b}H_{b-3/2}(2z)$$

with $a=1, b_1=3/2, b_2=b$, the transformation becomes

$$\frac{\pi}{4}\Gamma(\delta)\Gamma(\mu)x^{\frac{1}{2}-\delta}y^{\frac{1}{2}-\mu}\int_0^\infty t^{\sigma-\delta-\mu}e^{-zt}W_{k,m}(\beta t)L_{\delta-\frac{3}{2}}(2xt)H_{\mu-\frac{3}{2}}(2yt)\ dt$$

$$= \frac{\beta^{m+\frac{1}{2}}\Gamma(A')}{(z+\frac{\beta}{2})^A} \left\{ \frac{\Gamma(A)}{\Gamma(C)} F^{(3)} \left[\begin{array}{llllll} \frac{A}{2}, \frac{A+1}{2} :: & \frac{A'}{2}, \frac{A'+1}{2}; & -; & - : 1, -; 1, -; & \frac{B}{2}, & \frac{B+1}{2}; \\ \frac{C}{2}, \frac{C+1}{2} :: & & & : \frac{3}{2}, \delta; \frac{3}{2}, \mu; & \frac{1}{2} & ; \end{array} \right. \right.$$

$$\left. \frac{16x^2}{(2z+\beta)^2}, \frac{-16y^2}{(2z+\beta)^2}, \frac{(2z-\beta)^2}{(2z+\beta)^2} \right] + \left(\frac{2z-\beta}{2z+\beta} \right)$$

$$\times \frac{B\Gamma(A+1)}{\Gamma(C+1)} F^{(3)} \left[\begin{array}{llllll} \frac{A+1}{2}, \frac{A+2}{2} :: & \frac{A'}{2}, \frac{A'+1}{2}; & -; & - : 1, -; 1, -; & \frac{B}{2}, & \frac{B+1}{2}; \\ \frac{C+1}{2}, \frac{C+2}{2} :: & & & : \frac{3}{2}, \delta; \frac{3}{2}, \mu; & \frac{3}{2} & ; \end{array} \right.$$

$$\left. \left. \frac{16x^2}{(2z+\beta)^2}, \frac{-16y^2}{(2z+\beta)^2}, \frac{(2z-\beta)^2}{(2z+\beta)^2} \right] \right\}. \tag{3.5}$$

(vi). On taking $k = m + \frac{1}{2}$ in (2.1), we get

$$\int_0^\infty t^{\sigma-1} e^{-zt} W_{m+\frac{1}{2},m}(\beta t) {}_pF_q \left(\begin{array}{ll} a_1, a_2, ..., a_p; & \\ & x^2t^2 \\ b_1, b_2, ..., b_q; & \end{array} \right) {}_uF_v \left(\begin{array}{ll} \lambda_1, \lambda_2, ..., \lambda_u; & \\ & -y^2t^2 \\ \mu_1, \mu_2, ..., \mu_v; & \end{array} \right) dt$$

$$= \frac{\beta^{m+\frac{1}{2}}\Gamma(A')}{(z+\frac{\beta}{2})^A} \frac{\Gamma(A)}{\Gamma(C)}$$

$$F^{(3)} \left[\begin{array}{llll} \frac{A}{2}, \frac{A+1}{2} :: & \frac{A'}{2}, \frac{A'+1}{2}; -; - & : a_1, a_2, ..., a_p; & \lambda_1, \lambda_2, ..., \lambda_u; \\ \frac{A'}{2}, \frac{A'+1}{2} :: & & : b_1, b_2, ..., b_q; & \mu_1, \mu_2, ..., \mu_v; \end{array} \frac{16x^2}{(2z+\beta)^2}, \right.$$

$$\left. \frac{-16y^2}{(2z+\beta)^2}, \frac{(2z-\beta)^2}{(2z+\beta)^2} \right] \tag{3.6}$$

(vii). On taking $p = u = 1, q = v = 2$ in (2.1) we reduces to a well known result by W. A. Khan, N. U. Khan and M. Kamarujjama [9].

(viii). On taking $p = 0, q = 1, u = 1$ and $v = 2$ and having suitable parametric arrangements as $b_1 = 1 + \nu$, the transformation (2.1) reduces to

$$\int_0^\infty t^{\sigma-\nu-1} e^{-zt} W_{k,m}(\beta t) I_\nu(2xt) \, {}_1F_2 \left(\begin{array}{ll} \lambda_1, & -; \\ & & -y^2t^2 \\ \mu_1, & \mu_2; \end{array} \right) dt = \frac{x^\nu \beta^{m+\frac{1}{2}}\Gamma(A')}{\Gamma(\nu+1)(z+\frac{\beta}{2})^A} \left\{ \frac{\Gamma(A)}{\Gamma(C)} \right.$$

$$F^{(3)}\left[\begin{array}{l} \frac{A}{2},\frac{A+1}{2}::\ \frac{A'}{2},\frac{A'+1}{2};\ -;\ -:-;\ \lambda_1,-;\ \frac{B}{2},\ \frac{B+1}{2};\\ \frac{C}{2},\frac{C+1}{2}::\ :1+\nu;\ \mu_1,\mu_2;\ \frac{1}{2}\ ; \end{array} \frac{16x^2}{(2z+\beta)^2}, \frac{-16y^2}{(2z+\beta)^2},\frac{(2z-\beta)^2}{(2z+\beta)^2}\right]+\left(\frac{2z-\beta}{2z+\beta}\right)\frac{B\Gamma(A+1)}{\Gamma(C+1)}$$
$$F^{(3)}\left[\begin{array}{l} \frac{A+1}{2},\frac{A+2}{2}::\ \frac{A'}{2},\frac{A'+1}{2};\ -;\ -:-;\ \lambda_1,-;\ \frac{B+1}{2},\ \frac{B+2}{2};\\ \frac{C+1}{2},\frac{C+2}{2}::\ :1+\nu;\ \mu_1,\mu_2;\ \frac{3}{2}\ ; \end{array} \frac{16x^2}{(2z+\beta)^2},\frac{-16y^2}{(2z+\beta)^2},\frac{(2z-\beta)^2}{(2z+\beta)^2}\right]\Bigg\}, \tag{3.7}$$

where I_ν is the modified Bessel function of 1st kind [14], which is a known result of B. Khan and M. A. Pathan [5].

(ix). On taking $p=q=u=v=1$, $a_1=\lambda_1=a$ and $b_1=\mu_1=b$ in (2.1) and using Ramanujan's Theorem [15; p. 106 (5)]

$${}_1F_1\left[\begin{array}{ll}\alpha & ;\\ \beta & ;\end{array} x\right]\ {}_1F_1\left[\begin{array}{ll}\alpha & ;\\ \beta & ;\end{array} -x\right] = {}_2F_3\left[\begin{array}{llll}\alpha, & \beta-\alpha & & ;\\ \beta, & \frac{\beta}{2}, & \frac{1}{2}\beta+\frac{1}{2} & ;\end{array} \frac{x^2}{4}\right]$$

the transformation reduces to

$$\int_0^\infty t^{\sigma-1}e^{-zt}W_{k,m}(\beta t){}_2F_3\left(\begin{array}{llll} a, & b-a, & - & ;\\ b, & \frac{b}{2}, & \frac{b}{2}+\frac{1}{2} & ;\end{array} \frac{x^4t^4}{4}\right)dt = \frac{\beta^{m+\frac{1}{2}}\Gamma(A')}{(z+\frac{\beta}{2})^A}\left\{\frac{\Gamma(A)}{\Gamma(C)}\right.$$
$$F^{(3)}\left[\begin{array}{l} \frac{A}{2},\frac{A+1}{2}::\ \frac{A'}{2},\frac{A'+1}{2};\ -;\ -:\ a;\ a;\ \frac{B}{2},\ \frac{B+1}{2};\\ \frac{C}{2},\frac{C+1}{2}::\ :\ b;\ b;\ \frac{1}{2}\ ; \end{array} \frac{16x^2}{(2z+\beta)^2},\frac{-16x^2}{(2z+\beta)^2},\frac{(2z-\beta)^2}{(2z+\beta)^2}\right]$$
$$+\left(\frac{2z-\beta}{2z+\beta}\right)\frac{B\Gamma(A+1)}{\Gamma(C+1)}$$
$$F^{(3)}\left[\begin{array}{l} \frac{A+1}{2},\frac{A+2}{2}::\ \frac{A'}{2},\frac{A'+1}{2};\ -;\ -:\ a;\ a;\ \frac{B+1}{2},\ \frac{B+2}{2};\\ \frac{C+1}{2},\frac{C+2}{2}::\ :\ b;\ b;\ \frac{3}{2}\ ; \end{array} \frac{16x^2}{(2z+\beta)^2},\frac{-16x^2}{(2z+\beta)^2},\frac{(2z-\beta)^2}{(2z+\beta)^2}\right]\Bigg\}, \tag{3.8}$$

where ${}_2F_3$ is the generalized hypergeometric series given in (1.7).

(x). On taking $p=u=0$, $q=v=2$, $b_1=\mu_1=a$, $b_2=\mu_2=b$, $x=y$ in (2.1) and using the result [15; p. 106 (7)], we get

$$\int_0^\infty t^{\sigma-1}e^{-zt}W_{k,m}(\beta t)\,{}_3F_8\left(\begin{array}{c}\frac{1}{3}(a+b-1),\frac{1}{3}(a+b),\frac{1}{3}(a+b+1)\\ a,b,\frac{a}{2},\frac{a}{2}+\frac{1}{2},\frac{b}{2},\frac{b}{2}+\frac{1}{2},\frac{1}{2}(a+b-1),\frac{1}{2}(a+b)\end{array};\ \frac{-27x^4t^4}{64}\right)dt$$

$$=\frac{\beta^{m+\frac{1}{2}}\Gamma(A')}{(z+\frac{\beta}{2})^A}\left\{\frac{\Gamma(A)}{\Gamma(C)}\right.$$

$$F^{(3)}\left[\begin{array}{l}\frac{A}{2},\frac{A+1}{2}::\ \frac{A'}{2},\frac{A'+1}{2};\ -;\ -:-,-;-,-;\ \frac{B}{2},\ \frac{B+1}{2};\\ \frac{C}{2},\frac{C+1}{2}::\ :a,b;a,b;\ \frac{1}{2}\ ;\end{array}\ \frac{16x^2}{(2z+\beta)^2},\right.$$

$$+\left(\frac{2z-\beta}{2z+\beta}\right)\frac{B\Gamma(A+1)}{\Gamma(C+1)}$$

$$F^{(3)}\left[\begin{array}{l}\frac{A+1}{2},\frac{A+2}{2}::\ \frac{A'}{2},\frac{A'+1}{2};\ -;\ -:-,-;-,-;\ \frac{B+1}{2},\ \frac{B+2}{2};\\ \frac{C+1}{2},\frac{C+2}{2}::\ :a,b;a,b;\ \frac{3}{2}\ ;\end{array}\right.$$

$$\left.\left.\frac{16x^2}{(2z+\beta)^2},\frac{-16x^2}{(2z+\beta)^2},\frac{(2z-\beta)^2}{(2z+\beta)^2}\right]\right\}, \tag{3.9}$$

where ${}_3F_8$ is the generalized hypergeometric series given in (1.7).

4. Conclusion

In this paper, we have developed a new class of integral transforms involving the product of the Whittaker function and two generalized hypergeometric functions. These integrals were expressed in terms of finite sums of the Srivastava triple hypergeometric function, thereby yielding results of broad generality. Since Whittaker and generalized hypergeometric functions encompass many classical functions, the derived transforms provide a unified framework that subsumes and extends numerous known integral formulas. By suitable parameter choices, the results reduce to identities involving Laguerre and Hermite polynomials, the confluent hypergeometric function, and the Fox H-function, among others. This highlights both the flexibility and the unifying power of the obtained formulas. From the applications perspective, the findings enrich the theory of integral transforms in mathematical analysis, provide computational tools for problems in mathematical physics such as quantum mechanics and wave propagation, and offer potential applications in engineering models governed by special functions and fractional operators. Overall, this work not only generalizes existing results but also lays the foundation for further exploration of integral transforms associated with multivariable special functions and their applications across mathematics and the applied sciences. An interesting future direction would be to express the Whittaker function in terms of the Beta function and its matrix analogues. Developing such Beta or Beta-matrix representations of Whittaker functions could open new avenues for studying matrix-valued special functions, fractional operators, and applications in mathematical physics. For instance, in quantum mechanics, the radial Schrödinger equation for a particle in a Coulomb potential has solutions in terms of Whittaker functions. By expressing these solutions through Beta or Beta-matrix representations, one can derive new integral forms and recurrence relations that may simplify the analysis of bound states and scattering problems.

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