J. of Ramanujan Society of Mathematics and Mathematical Sciences
Vol. 13, No. 1 (2025), pp. 19-38

DOI: 10.56827/JRSMMS.2025.1301.2 **ISSN (Online): 2582-5461**

ISSN (Print): 2319-1023

APPROXIMATION, EXISTENCE AND UNIQUENESS OF THE INTEGRABLE LOCAL SOLUTION OF NONLINEAR HYBRID FUNCTIONAL INTEGRAL EQUATIONS

Janhavi B. Dhage and Bapurao C. Dhage

Kasubai, Gurukul Colony,
Ahmedpur - 413515, Latur, Maharashtra, INDIA

E-mail : jbdhage@gmail.com, bcdhage@gmail.com

(**Received:** *Jul. 03, 2025* **Accepted:** *Nov. 30, 2025* **Published:** *Dec. 30, 2025)*

Abstract: In this paper, we prove a couple of approximation results for existence and uniqueness of the integrable local solutions of nonhomogeneous nonlinear hybrid functional integral equations under weaker partial compactness, partial Lipschitz and usual monotonicity type conditions. We employ the Dhage monotone iteration method based on the recent hybrid fixed point theorems of Dhage (2024) while establishing our main results of this paper. Our hypotheses and abstract results are also illustrated with a couple of numerical examples.

Keywords and Phrases: Functional integral equation, Hybrid fixed point principle, Dhage iteration method; Approximation result, Existence and uniqueness theorem.

2020 Mathematics Subject Classification: Primary 45G10, Secondary 47H10.

1. Introduction

Theoretical approximation results for existence and uniqueness of continuous and integrable local solutions for nonlinear differential and integral equations can be obtained under usual Lipschitz condition on the nonlinearity or monotonicity condition blending with the existence of upper and lower solutions of the related nonlinear problems. These results are achieved by the applications of Banach fixed point theorem or by monotone iteration method given in Ladde *et al.* [25] or generalized iteration method as depicted in Hekkila and Lakshmikantham [26].

We observe that the hypotheses of continuity, boundedness and monotonicity of the nonlinearity are natural, but Lipschitzicity and existence of lower and upper solutions are stringent conditions which are rather difficult to hold for most of the nonlinear equations. Therefore, it is of interest to obtain such approximation results under weaker conditions or without the requirement of upper and lower solutions which is the main motivation of the present paper. The novelty of he paper lies in the new approach and novel application of a recent hybrid fixed point theorem Dhage [11]. In the present study we obtain approximation results for existence and uniqueness of integrable solutions of a certain nonlinear hybrid functional integral equations via Dhage iteration method. However, the considered problem of this paper can be explored to obtain local approximate stability and numerical analysis of the integrable solution as done in Dhage *et al.* [18].

Given a closed and bounded interval $J = [0, T]$ in $\mathbb{R}$, the set of real numbers, we consider a nonlinear hybrid functional integral equation (in short HFIE)

$$x(t) = q(t) + \lambda \int_0^{\sigma(t)} k(t, s) f(s, x(s))\, ds, \;\; t \in J, \tag{1}$$

where $\lambda \in \mathbb{R}_+ = (0, \infty)$, $\sigma : J :\to J$ is continuous and the functions $q : J :\to \mathbb{R}$, $k : J \times J :\to \mathbb{R}$, $f : J \times \mathbb{R} \to \mathbb{R}$ satisfy certain hybrid conditions, that is, mixed conditions of "compactness, Lipschitz and monotonicity" to be specified later.

Definition 1. *By an integrable solution of the nonlinear HFIE (1) we mean a function $x \in L^1(J, \mathbb{R})$ that satisfies the equation (1) defined on J, where $L^1(J, \mathbb{R})$ is the space of Lebesgue integrable Real-valued functions on J. Furthermore, if a solution x of the HFIE (1) lies in the neighborhood of a point $x_0 \in L^1(J, \mathbb{R})$, we say it is a local or neighborhood solution of the HFIE (1) defined on J.*

Remark 1. *The concept of local or neighborhood solution of the HFIE (1) is different from that of usual notion of local solution as mentioned in Coddington* [4]*. In the terminology of Coddington* [4]*, it is a nonlocal solution of the HFIE (1) defined on all of J. Similarly, by a classical solution we generally mean a function $x \in C(J, \mathbb{R})$ that satisfies the given HFIE (1), where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J. Note that every classical solution is a integrable solution, but the converse may not be true.*

The HFIE (1) is a nonlinear functional integral equation of second type and includes as special cases the nonlinear Volterra integral equation of second type,

$$x(t) = q(t) + \lambda \int_0^{t} k(t, s) f(s, x(s))\, ds, \;\; t \in J, \tag{2}$$

provided $\sigma(t) = t$ and nonlinear Fredholm integral equation of second type

$$x(t) = q(t) + \lambda \int_0^T k(t,s) f(s, x(s))\, ds, \;\; t \in J, \tag{3}$$

provided $\sigma(t) = T$. Therefore, the results of the problem (1) includes the approximate local existence and local uniqueness of the integrable solutions for the hybrid Volterra and Fredholm integral equations (2) and (3) as particular cases. It is needless to say the importance of the HIE (2) and (3) and consequently HFIE (1), because such integral equations appear in several biological and physical situations as mentioned in Banas [2], Mydlarczyk [28], Raffoul [29] and references therein. As the nonlinearity or nonlinearities involved in a hybrid differential and integral equations satisfy certain characteristics from different subject of mathematics like algebra, analysis and topology, the existence results of such hybrid equations are very much useful for determining the qualitative behavior of the related dynamical systems in a better way. The HFIE (1) has been studied very extensively in the literature for different aspects of the solution using different techniques from algebra, analysis and topology. Here, we discuss the HFIE (1) for approximation of the local solution via Dhage iteration method, especially we discuss the existence and uniqueness of integrable local solution by method of successive approximations using Dhage iteration method embedded in the recent hybrid fixed point theorems of Dhage [10, 11].

2. Preliminaries

We place the problem of HFIE (1) in the function space $L^1(J, \mathbb{R})$ of Lebesgue integrable real-valued functions defined on the interval J. Now, we introduce a norm $\|\cdot\|_{L^1}$ in $L^1(J,\mathbb{R})$ defined by

$$\|x\|_{L^1} = \int_0^T |x(t)|\, dt, \tag{4}$$

and an order relation $\preceq$ in $L^1(J,\mathbb{R})$ by the cone K given by

$$K = \{x \in L^1(J,\mathbb{R}) \mid x(t) \geq 0 \;\; \text{a.e.} \;\; t \in J\}. \tag{5}$$

Thus,

$$x \preceq y \iff y - x \in K,$$

or equivalently,

$$x \preceq y \iff x(t) \leq y(t) \;\; \text{a. e.} \;\; t \in J. \tag{6}$$

The details of order cones and related order relations may be found in Deimling [5], Guo and Lakshmikantham [23] and references therein. It is known that the Banach space $L^1(J,\mathbb{R})$ together with the order relations $\preceq$ becomes an ordered Banach space which we denote for convenience, by $\big(L^1(J,\mathbb{R}),K\big)$. We denote the open and closed spheres centered at $x_0 \in L^1(J,\mathbb{R})$ of radius r, by

$$B_r(x_0) = \{x \in L^1(J,\mathbb{R}) \mid \|x - x_0\|_{L^1} < r\} = B(x,r),$$

and

$$B_r[x_0] = \{x \in L^1(J,\mathbb{R}) \mid \|x - x_0\|_{L^1} \leq r\} = \overline{B(x,r)}, \tag{7}$$

respectively. It is clear that $B_r[x_0] = \overline{B_r(x_0)}$.

We need the following result concerning the compactness of a subset in $L^1(J,\mathbb{R})$ in what follows.

Lemma 1. (Kolmogorov compactness criterion [21]) *Let $\Omega \subseteq L^p(J,\mathbb{R})$, $1 \leq p < \infty$. If*

(i) Ω is bounded, and

(ii) $x_\eta \to x$ as $\eta \to 0$ uniformly w.r.t. $x \in \Omega$, where $x_\eta(t) = \dfrac{1}{\eta}\displaystyle\int_t^{t+\eta} x(s)\,ds.$

Then Ω is a relatively compact subset of $L^p(J,\mathbb{R})$.

It is well-known that the fixed point as well as hybrid fixed point theoretic techniques are very much useful in the subject of nonlinear analysis for dealing with the nonlinear equations qualitatively, see Granas and Dugundji [22], Raffoul [29] and the references therein. Here, we employ the Dhage monotone iteration method or simply **Dhage iteration method** based on the generalizations two hybrid fixed point theorems in the partially ordered abstract spaces. Generalizing the hybrid fixed point theorem of Dhage [11] and Dhage *et al.* [13], the present second author in [11] proved a Schauder type hybrid fixed point theorem in a partially ordered Banach space. Before stating this theorem, we give some preliminaries needed in the sequel.

Let $\big(E, d, \preceq\big)$ be a partially ordered metric space. Two elements x and y of E are said to be **comparable** if either $x \preceq y$ or $y \preceq x$ holds. A subset C of E is called a **chain** if all the elements of C are comparable. A subset S of E is called **regular** if a monotone nondecreasing (resp. monotone nonincreasing) sequence $\{x_n\}$ in E converges to x_*, then $x_n \preceq x_*$ (resp. $x_* \preceq x_n$) for all $n \in \mathbb{N}$. The metric d and the order relation $\preceq$ are said to be **compatible** in S if a monotone sequence $\{x_n\}$ in S has a convergent subsequence, then the original sequence $\{x_n\}$ is convergent and converges to the same limit point. S is called a **Janhavi set** if d and $\preceq$ are compatible in it. S is called **partial bounded** (resp. partially closed, partially compact) if every chain C in S is bounded (resp. closed, compact). A few details of the above notions appear in Dhage [7, 8, 9].

A mapping $\mathcal{T} : S \to S$ is called **monotone nondecreasing** (resp. monotone nonincreasing) if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ (resp. $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$). $\mathcal{T}$ is **monotone** if it is either monotone nondereasing or monotone nonincreasing. $\mathcal{T}$ is called **partial bounded** (resp. partially totally bounded or partially precompact) if $\mathcal{T}(S)$ is partially bounded (resp. partially totally bounded or partially precompact for partially bounded S). T is **partially continuous** if and only if $\{x_n\} \subset S$ converges to x with $x_n \preceq x$ or $x \preceq x_n$ for each $n \in \mathbb{N}$, then $\mathcal{T}x_n \to \mathcal{T}x$ for each $x \in S$. T is called **partial completely continuous** if it is partially continuous and partially totally bounded. The details of above concepts along with applications appear in Dhage [10, 11]. The following lemma is frequently used in the hybrid fixed point theory and its applications to nonlinear differential and integral equations.

Lemma 2. *Every mapping $\mathcal{T}$ on a subset S of an ordered metric space E into itself is partially continuous if and only if it is continuous on every chain C of S.*

Proof. The proof is given in Dhage and Dhage [16], but for the sake of completeness, we give the details of the proof. Suppose that $\mathcal{T}$ is a partially continuous mapping on S into itself. Let $\{x_n\}$ be an arbitrary sequence of points in a chain C converging to a point $x \in C$. Then either $x_n \preceq x$ or $x \preceq x_n$ for each $n \in \mathbb{N}$. From the partial continuity of $\mathcal{T}$ it follows that $\mathcal{T}x_n \to \mathcal{T}x$ as $n \to \infty$. As a result $\mathcal{T}$ is a continuous mapping on C. Conversely suppose that $\mathcal{T}$ is continuous on every chain C of S. Let $x \in S$ be arbitrary and let $\{x_n\}$ be any sequence of points in S converging to a point $x \in S$ with $x_n \preceq x$ or $x \preceq x_n$ for each $n \in \mathbb{N}$. Then $C = \{x_n\} \bigcup \{x\}$ is a chain in S. By continuity of $\mathcal{T}$ in every C, we obtain $\mathcal{T}x_n \to \mathcal{T}x$ as $n \to \infty$, showing that $\mathcal{T}$ is a partially continuous on S. This completes the proof.

Now we are equipped with all the necessary details to state our required hybrid fixed point theorems needed in what follows.

Theorem 1. *Let S be a non-empty, partial closed and partial bounded subset of a regular partially ordered Banach space $\big(E, \|\cdot\|, \preceq\big)$. Suppose that $\mathcal{T} : S \to S$ is a partial completely continuous and monotone nondecreasing operator and every chain C in $\mathcal{T}(S)$ is Janhavi set. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* in S and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to x^*.*

Proof. The proof is similar to a hybrid fixed point theorem proved in Dhage [10] with obvious modifications, however the details appear in Dhage [11].

Theorem 2. (Dhage [10]) *Let $B_r[x]$ denote the partial closed ball centered at x of radius r, in a regular partially ordered Banach space $\big(E, \|\cdot\|, \preceq,\big)$ and let $\mathcal{T} : E \to E$ be a monotone nondecreasing and partial contraction operator with contraction constant q. If there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$ satisfying*

$$\|x_0 - \mathcal{T}x_0\| \leq (1-q)r, \tag{8}$$

for some real number $r > 0$, then $\mathcal{T}$ has a unique comparable fixed point x^* in $B_r[x_0]$ and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive iterations converges monotonically to x^*. Furthermore, if every pair of elements in X has a lower or upper bound, then x^* is unique.

Remark 2. *We note that every pair of elements in a partially ordered set (in short poset)* (**poset**) *$(E, \preceq)$ has a lower or upper bound if $(E, \preceq)$ is a lattice, that is, $\preceq$ is a lattice order in E. In this case the poset $(E, \|\cdot\|, \preceq)$ is called a* **partially lattice ordered Banach space**. *There do exist several lattice partially ordered Banach spaces which are useful for applications in nonlinear analysis. For example, every Banach lattice is a partially lattice ordered Banach space. Notice that, $L^1(J, \mathbb{R})$ is a partially lattice ordered Banach space which is a complete lattice (see Dhage* [6]*). The details of the lattice structure of a Banach space appear in the monograph Birkhoff* [3].

As a consequence of Remark 2, we obtain.

Theorem 3. *Let $B_r[x]$ denote the partial closed ball centered at x of radius r for some real number $r > 0$, in a regular partially lattice ordered Banach space $\big(E, \|\cdot\|, \preceq,\big)$ and let $\mathcal{T} : E \to E$ be a monotone nondecreasing and partial contraction operator with contraction constant q. If there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$ satisfying (8), then $\mathcal{T}$ has a unique fixed point ξ^* in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

If a Banach space X is partially ordered by an order cone K in X, then in this case we simply say X is an **ordered Banach space** which we denote by (X, K). Similarly, if an ordered Banach space (X, K), where the partial order $\preceq$ defined by the con K is a lattice order, then (X, K) is called the **lattice ordered Banach space**. Clearly, an ordered Banach space $\big(L^1(J, \mathbb{R}), K\big)$ of Lebesgue integrable real-valued functions defined on the closed and bounded interval J is a lattice ordered Banach space, where the cone K is given by $K = \{x \in L^1(J, \mathbb{R}) \mid x \succeq 0 \ \text{a. e. on } J\}$. The details of the cones and their properties appear in Guo and Lakshmikantham [23]. Then, we have the following useful results concerning the ordered Banach spaces proved in Dhage [8, 9].

Lemma 3. (Dhage [8, 9]) *Every ordered Banach space (X, K) is regular.*

Lemma 4. (Dhage [8,9]) *Every partially compact subset S of an ordered Banach space (X, K) is a Janhavi set in X.*

As a consequence of Lemmas 3 and 4, we obtain the following applicable hybrid fixed point theorems which we need in what follows.

Theorem 4. *Let S be a non-empty, partially closed and partially bounded subset of an ordered Banach space (X, K) and let $\mathcal{T} : S \to S$ be a partially completely continuous and monotone nondecreasing operator. If there exists an element $x_0 \in S$ such that $x_0 \preceq Tx_0$ or $x_0 \succeq Tx_0$, then $\mathcal{T}$ has a fixed point $\xi^* \in S$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

Theorem 5. *Let $B_r[x]$ denote the partial closed ball centered at x of radius r for some real number $r > 0$, in a lattice ordered Banach space $\big(X, K\big)$ and let $\mathcal{T} : (X, K) \to (X, K)$ be a monotone nondecreasing and partial contraction operator with contraction constant q. If there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T} x_0$ or $x_0 \succeq \mathcal{T} x_0$ satisfying (8), then $\mathcal{T}$ has a unique fixed point ξ^* in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

A few details of hybrid fixed point theorems and related applications appear in Deimling [5], Dhage [7, 8, 9], Dhage and Dhage [12], Dhage *et al.* [13, 17, 18], Dhage and Dhage [14, 15], Ardjouli and Djoudi [1], Gupta *et al.* [24], You and Han [32] and references therein.

3. Local Approximation Results

We consider the following definitions in the sequel.

Definition 2. *A function $f : J \times \mathbb{R} \to \mathbb{R}$ is said to be* $L^1_{\mathbb{R}}$-**Carathéodory** *if*

(i) the map $t \mapsto f(t, x)$ is measurable for each $x \in \mathbb{R}$,

(ii) the map $x \mapsto f(t, x)$ is continuous for almost everywhere $t \in J$, and

(iii) there exists a function $h \in L^1(J, \mathbb{R})$ such that

$$|f(t, x)| \leq h(t) \quad a.e. \quad t \in J,$$

for all $x \in \mathbb{R}$.

Definition 3. *A function $k : J \times J \to \mathbb{R}$ is said to satisfy* ***integrability condition*** *if*

(i) the map $(t, s) \mapsto k(t, s)$ is jointly measurable, and

(iii) there exists a function $\gamma_k \in L^1(J, \mathbb{R})$ *such that*

$$|k(t,s)| \leq \gamma_k(s) \quad a.\,e.\ t, s \in J.$$

Lemma 5. (Granas and Dugundji [22]) *If* $f(t,x)$ *is* $L^1_{\mathbb{R}}$*-Carathéodory, then the function* $t \mapsto f(t, x(t))$ *is measurable and Lebesgue integrable for each* $x \in L^1(J, , \mathbb{R})$*.*

Lemma 6. (Krasnoselskii [26]) *If the function* $f : J \times \mathbb{R} \to \mathbb{R}$ *is* $L^1_{\mathbb{R}}$*-Carathéodory, then the superposition operator* F *defined by* $(Fx)(t) = f(t, x(t))$ *maps continuously the space* $L^1(J, \mathbb{R})$ *into itself.*

Remark 3. *Note that if the function* $k(t,s)$ *is* $L^1_{\mathbb{R}}$*-Carathéodory, then it satisfies the integrability condition on* $J \times J$*.*

We need the following set of hypotheses in what follows.

(H_0) The function $q : J \to \mathbb{R}$ is Lebesgue integrable.

(H_1) The function k is nonnegative and satisfies integrability conditions on $J \times J$. Moreover, there is a real number $c > 0$ such that $\gamma_k(t) \leq c$ for almost everywhere $t \in J$.

(H_2) There exists a constant $\alpha > 0$ such that

$$0 \leq f(t,x) - f(t,y) \leq \alpha(x-y) \quad \text{a. e.} \quad t \in J,$$

for all $x, y \in \mathbb{R}$ with $x \geq y$, where $c\,\lambda\,\alpha\,T < 1$, where c is given in hypothesis (H_1).

(H_3) The function f is $L^1_{\mathbb{R}}$-Carathéodory.

(H_4) $f(t,x)$ is nondecreasing in x for almost everywhere on J.

(H_5) $f(t, q(t)) \geq 0$ a. e. $t \in J$, where the function q is given in hypothesis (H_0).

Theorem 6. *Suppose that the hypotheses* (H_0), (H_1), (H_3), (H_4) *and* (H_5) *hold. If there exists a real number* $r > 0$ *such that* $c\lambda \|h\|_{L^1} T \leq r$*, then the HFIE (1) has an integrable solution* x^* *in* $B_r[x_0]$*, where,* $x_0 \equiv q$*, and the sequence* $\{x_n\}_{n=0}^{\infty}$ *of successive approximations defined by*

$$\left.\begin{aligned} x_0(t) &= q(t), \ \ t \in J, \\ x_{n+1}(t) &= q(t) + \lambda \int_0^{\sigma(t)} k(t,s) f(s, x_n(s))\, ds, \ \ t \in J, \end{aligned}\right\} \tag{9}$$

where $n = 0, 1, \ldots$; *is monotone nondecreasing and converges to* x^*.

Proof. Set $X = L^1(J, \mathbb{R})$. Clearly, X is an ordered Banach space w.r.t. the norm $\|\cdot\|_{L^1}$ and the order relation $\preceq$ given by (4) and (6) respectively. Let x_0 be an initial function on J such that $x_0(t) = q(t)$ a. e. $t \in J$ and define a closed ball $B_r[x_0]$ in X defined by (7), where the number r satisfies the inequality $c\lambda \|h\|_{L^1} T \leq r$. Now, define an operator $\mathcal{T}$ on $B_r[x_0]$ into X by

$$\mathcal{T}x(t) = q(t) + \lambda \int_0^{\sigma(t)} k(t,s) f(s, x(s))\, ds, \;\; t \in J. \tag{10}$$

Clearly, the integral and the consequently the operator $\mathcal{T}$ given in (10) is well defined in view of Lemma 5. Then the HFIE (1) is transformed into a hybrid operator equation (HOE),

$$\mathcal{T}x(t) = x(t), \;\; t \in J. \tag{11}$$

We shall show that the operator $\mathcal{T}$ satisfies all the conditions of Theorem 5 on $B_r[x_0]$ in the following series of steps.

Step I: The operator $\mathcal{T}$ maps $B_r[x_0]$ into itself.

Let $x \in B_r[x_0]$ be arbitrary. Then,

$$\begin{aligned} |\mathcal{T}x(t) - x_0(t)| &= \lambda \left| \int_0^{\sigma(t)} k(t,s) f(s, x(s))\, ds \right| \\ &\leq \lambda \int_0^{\sigma(t)} |k(t,s)| \left| f(s, x(s)) \right| ds \\ &\leq c\lambda \int_0^T h(s)\, ds \\ &= c\lambda \|h\|_{L^1}. \end{aligned}$$

Taking the integral on both sides from 0 to T w.r.t. t, we obtain

$$\int_0^T \left|(\mathcal{T}x - x_0)(t)\right| dt \leq c\lambda \int_0^T \|h\|_{L^1}\, dt = c\lambda \|h\|_{L^1} T.$$

Therefore,

$$\|\mathcal{T}x - q\|_{L^1} \leq c\lambda \|h\|_{L^1} T \leq r.$$

This implies that $\mathcal{T}x \in B_r[x_0]$ for all $x \in B_r[x_0]$.

Step II: $\mathcal{T}$ is a monotone nondecreasing operator on $B_r[x_0]$.

Let $x, y \in B_r[x_0]$ be any two elements such that $x \succeq y$ almost everywhere on J. Then by (H$_4$) we obtain

$$\begin{aligned}\mathcal{T}x(t) &= q(t) + \lambda \int_{t_0}^{\sigma(t)} k(t,s) f(s, x(s))\, ds \\ &\geq q(t) + \lambda \int_{t_0}^{\sigma(t)} k(t,s) f(s, y(s))\, ds \\ &= \mathcal{T}y(t),\end{aligned}$$

for almost every $t \in J$. So, $\mathcal{T}x \succeq \mathcal{T}y$ almost everywhere on J– that is, $\mathcal{T}$ is monotone nondecreasing on $B_r[x_0]$.

Step III: $\mathcal{T}$ is a partially continuous operator on $B_r[x_0]$.

Let C be a chain in $B_r[x_0]$ and let $\{x_n\}$ be a sequence in C converging almost everywhere to a point $x \in C$. Then by Lebesgue dominated convergence theorem, we have

$$\begin{aligned}\lim_{n\to\infty} \mathcal{T}x_n(t) &= \lim_{n\to\infty} \left[q(t) + \lambda \int_{t_0}^{\sigma(t)} k(t,s) f(s, x_n(s))\, ds \right] \\ &= q(t) + \lambda \lim_{n\to\infty} \int_{t_0}^{\sigma(t)} k(t,s) f(s, x_n(s))\, ds \\ &= q(t) + \lambda \int_{t_0}^{\sigma(t)} k(t,s) \Big[\lim_{n\to\infty} f(s, x_n(s)) \Big]\, ds \\ &= q(t) + \lambda \int_{t_0}^{\sigma(t)} k(t,s) f(s, x(s))\, ds \\ &= \mathcal{T}x(t),\end{aligned}$$

for almost every $t \in J$. Therefore, $\mathcal{T}x_n \to \mathcal{T}x$ pointwise on J.

Next, we show that $\mathcal{T}x_n$ converges uniformly to $\mathcal{T}x$ in $L^!(J, \mathbb{R})$. Now, $\{\mathcal{T}x_n\}$ is a sequence of Lebesgue integrable functions, so it is also a sequence of measurable functions on J. Similarly, $\mathcal{T}x$ is also a measurable function on J. Moreover, $|\mathcal{T}x_n(t)| \leq \|q\|_{L^1} + c\lambda\, \|h\|_{L^1}$ and $|\mathcal{T}x(t)| \leq \|q\|_{L^1} + c\lambda\, \|h\|_{L^1}$. Therefore, $\mathcal{T}x_n - \mathcal{T}x$ is measurable and

$$|\mathcal{T}x_n(t) - \mathcal{T}x(t)| \leq |\mathcal{T}x_n(t)| + |\mathcal{T}x(t)| \leq 2\big[\|q\|_{L^1} + c\lambda\, \|h\|_{L^1}\big],$$

for almost every $t \in J$. Now, by definition of the norm $\|\cdot\|_{l^1}$, we obtain

$$\|\mathcal{T}x_n - \mathcal{T}x\|_{L^1} = \int_0^T |\mathcal{T}x_n(t) - \mathcal{T}x(t)|\, dt.$$

Again, applying the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}\lim_{n\to\infty}\|\mathcal{T}x_n-\mathcal{T}x\|_{L^1}&=\lim_{n\to\infty}\int_0^T|\mathcal{T}x_n(t)-\mathcal{T}x(t)|\,dt\\&=\int_0^T\Big[\lim_{n\to\infty}|\mathcal{T}x_n(t)-\mathcal{T}x(t)|\Big]\,dt\\&\to 0\quad\text{as}\quad n\to\infty.\end{aligned}$$

This shows that $\mathcal{T}x_n\to\mathcal{T}x$ uniformly. As a result $\mathcal{T}$ is a partially continuous operator on $B_r[x_0]$ into itself. We mention that the partial continuity of the operator $\mathcal{T}$ can also be obtained by giving different arguments and by using Lemma 6 as done in Banas [2], Emmanuel [21] and Krasnoselsdkii [26].

Step IV: $\mathcal{T}$ is a partial compact operator on $B_r[x_0]$ into itself.

To show $\mathcal{T}$ is a partial compact operator, it is enough to prove that $\mathcal{T}(B_r[x_0])$ is a partially compact subset of $B_r[x_0]$. Let C be a chain in $\mathcal{T}(B_r[x_0])$. We show that $\mathcal{T}(C)$ is a relatively compact subset of $B_r[x_0]$. We apply the Kolomogorov theorem for compactness of a set in $L^1(J,\mathbb{R})$. Firstly, let $y\in\mathcal{T}(C)$ be any element. Then there is an element $x\in C$ such that $y=\mathcal{T}x$. Now, by hypothesis (H_3), we obtain

$$\|y\|_{L^1}\le\|q\|_{L^1}+\lambda\int_0^T\left|\int_0^{\sigma(t)}|k(t,s)|\,|f(s,x(s))|\,ds\right|\,dt\le\|q\|_{L^1}+c\lambda\|h\|_{L^1}T,$$

for all $y\in\mathcal{T}(C)$. This shows that $\mathcal{T}(C)$ is uniformly bounded subset of $L^1(J,\mathbb{R})$. Next, we show that $(\mathcal{T}x)_\eta\to\mathcal{T}x$ as $\eta\to 0$ uniformly for every $x\in C$. Now,

$$\begin{aligned}\|(\mathcal{T}x)_\eta-\mathcal{T}x\|_{L^1}&=\int_0^T\big|(\mathcal{T}x)_\eta(t)-\mathcal{T}x(t)\big|\,dt\\&=\int_0^T\left|\frac{1}{\eta}\int_t^{t+\eta}\big[\mathcal{T}x(s)\,ds-\mathcal{T}x(t)\big]\,ds\right|\,dt\\&\le\int_0^T\frac{1}{\eta}\int_t^{t+\eta}\big|\mathcal{T}x(s)-\mathcal{T}x(t)\big|\,ds\,dt.\end{aligned}\tag{12}$$

Since $\mathcal{T}x\in L^1(J,\mathbb{R})$, using the arguments that given in Swartz [29] (also see El-Sayed and Al-Issa [19]), we have

$$\frac{1}{\eta}\int_t^{t+\eta}\big|\mathcal{T}x(s)-\mathcal{T}x(t)\big|\,ds\to 0\quad\text{as}\quad\eta\to 0,$$

uniformly for $x \in C$. Substituting the above estimate in (11), we obtain

$$\|(\mathcal{T}x)_\eta - \mathcal{T}x\|_{L^1} \to 0 \quad \text{as} \quad \eta \to 0,$$

uniformly for $x \in C$. Therefore, $(\mathcal{T}x)_\eta \to \mathcal{T}x$ uniformly as $\eta \to 0$ for all $x \in C$. Now by an application of Kolomogorov theorem, we infer that $\mathcal{T}(C)$ is relatively compact subset of $B_r[x_0]$. Consequently, $\mathcal{T}$ is a partially compact operator on $B_r[x_0]$ into itself.

Step V: The element $x_0 \equiv q \in B_r[x_0]$ satisfies the order relation $x_0 \preceq \mathcal{T}x_0$ almost everywhere on J.

Since (H_5) holds, one has

$$\begin{aligned} x_0(t) &= q(t) + \lambda \int_{t_0}^{\sigma(t)} k(t,s) f(s, x_0(s))\, ds \\ &\le x_0(t) + \lambda \int_{t_0}^{\sigma(t)} k(t,s) f(s, q(t))\, ds \\ &= q(t) + \lambda \int_{t_0}^{\sigma(t)} k(t,s) f(s, x_0(s))\, ds \\ &= \mathcal{T}x_0(t), \end{aligned}$$

for almost every $t \in J$. As a result, we have $x_0 \preceq \mathcal{T}x_0$ almost everywhere on J. This shows that the initial function x_0 in $B_r[x_0]$ serves to satisfy the operator inequality $x_0 \preceq \mathcal{T}x_0$.

Thus, the operator $\mathcal{T}$ satisfies all the conditions of Theorem 4, and so $\mathcal{T}$ has a fixed point x^* in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotone nondecreasingly to x^* almost everywhere on J. This further implies that the HIE (1) and consequently the HFIE (1) has a integrable local solution x^* and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (9) converges monotone nondecreasingly to x^*. This completes the proof.

Next, we prove an approximation result for existence and uniqueness of the solution simultaneously under weaker form of one sided or partial Lipschitz condition.

Theorem 7. *Suppose that the hypotheses (H_0),(H_1), (H_2), (H_4) and (H_5) hold. Furthermore, if*

$$c\lambda\, \|h\|_{L^1} T \le (1 - c\lambda\, \alpha T) r, \quad c\lambda\, \alpha T < 1, \tag{13}$$

for some real number $r > 0$, then the HFIE (1) has a unique integrable local solution x^* in $B_r[x_0]$ defined on J, where $x_0 \equiv q$ almost everywhere on J and the *sequence* $\{x_n\}_{n=0}^{\infty}$ *of successive approximations defined by (9) converges monotone nondecreasingly to* x^*.

Proof. Set $(X, K) = \big(L^1(J,\mathbb{R}), \preceq\big)$, which is a lattice w. r. t. the lattice operations *meet* ($\wedge$) and *join* ($\vee$) defined by $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$ respectively, and so every pair of elements of X has a lower and an upper bound. Let x_0 be an initial function on J such that $x_0(t) = q(t)$ for almost everywhere $t \in J$ and consider the closed ball $B_r[x_0]$ centered at $x_0 \in L^1(J,\mathbb{R})$ of radius r, in the lattice ordered Banach space (X, K).

Define an operator $\mathcal{T}$ on X into X by (10). Clearly, $\mathcal{T}$ is monotone nondecreasing on X. To see this, let $x, y \in X$ be two elements such that $x \succeq y$ almost everywhere on J. Then, by hypothesis (H_4), we have

$$\mathcal{T}x(t) - \mathcal{T}y(t) = \lambda \int_0^{\sigma(t)} k(t,s)\big[f(s,x(s)) - f(s,y(s))\big]\,ds \geq 0,$$

for almost every $t \in J$. Therefore, $\mathcal{T}x \succeq \mathcal{T}y$, and consequently $\mathcal{T}$ is monotone nondecresing on X.

Next, we show that $\mathcal{T}$ is a partial contraction on X. Let $x, y \in X$ be such that $x \succeq y$. Then, by hypothesis (H_2), we obtain

$$\begin{aligned}
|\mathcal{T}x(t) - \mathcal{T}y(t)| &= \left|\lambda \int_{t_0}^{\sigma(t)} k(t,s)\big[f(s,x(s)) - f(s,y(s))\big]\,ds\right| \\
&\leq \lambda\alpha \left|\int_{t_0}^{\sigma(t)} k(t,s)\big(x(s) - y(s)\big)\,ds\right| \\
&\leq \lambda\alpha \int_{t_0}^{T} \gamma_k(s)|x(s) - y(s)|\,ds \\
&\leq c\lambda\,\alpha\,\|x - y\|_{L^1},
\end{aligned}$$

for almost every $t \in J$. Taking the integral from 0 to T on both sides of the above inequality yields

$$\|\mathcal{T}x - \mathcal{T}y\|_{L^1} \leq c\lambda\,\alpha\,T\,\|x - y\|_{L^1}, \quad c\lambda\,\alpha\,T < 1,$$

for all comparable elements $x, y \in X$. This shows that $\mathcal{T}$ is a partial contraction on X with contraction constant $\lambda\,k\,T$. Furthermore, it can be shown, as in the proof of Theorem 6, that the element $x_0 \in B_r[x_0]$ satisfies the relation $x_0 \preceq \mathcal{T}x_0$

in view of hypothesis (H$_4$). Finally, by hypothesis (H$_1$), one has

$$\begin{aligned}|x_0(t) - \mathcal{T}x_0(t)| &= |q(t) - \mathcal{T}q(t)| \\ &= \left|\lambda \int_0^{\sigma(t)} k(t,s)f(s,q(s))\,ds\right| \\ &\leq \lambda \int_0^T |k(t,s)|\,|f(s,q(s))|\,ds \\ &\leq c\lambda \int_0^T h(s)\,ds \\ &= c\lambda\,\|h\|_{L^1},\end{aligned}$$

for almost every $t \in J$. Now, from condition (12), we get

$$\begin{aligned}\|x_0 - \mathcal{T}x_0\|_{L^1} &= \int_0^T |x_0(t) - \mathcal{T}x_0(t)|\,dt \\ &\leq \int_0^T c\lambda\,\|h\|_{L^1}\,dt \\ &= c\lambda\,\|h\|_{L^1}T \\ &\leq \big(1 - c\lambda\,\alpha\,T\big)r,\end{aligned}$$

which shows that the condition (8) of Theorem 5 is satisfied. Hence, $\mathcal{T}$ has a unique fixed point x^* in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotone nondecreasingly to x^*. This further implies that the HOE (10) and consequently the HFIE (1) has a unique integrable local solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (9) converges monotone nondecreasingly to x^*. This completes the proof.

Remark 4. *The conclusion of Theorems 6 and 7 also remains true if we replace the hypothesis (H$_5$) with the following one.*

(H$'_5$) The function f satisfies inequality $f(t,q(t)) \leq 0$ a. e. $t \in J$, where the function q given in hypothesis (H$_1$).

In this case, the HFIE (1) has a integrable local solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (9) is monotone nonincreasing and converges to x^*.

Remark 5. *If the initial condition in the equation (1) is such that $q(t) > 0$ a. e. $t \in J$, then under the conditions of Theorem 6, the HFIE (1) has a integrable local* *positive solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (9) converges monotone nondecreasingly to the positive solution x^*. Similarly, under the conditions of Theorem 7, the HFIE (1) has a unique integrable local positive solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (9) converges monotone nondecreasingly to x^*.*

Example 1. Let $J = [0, 1] \subset \mathbb{R}$ and consider the HFIE

$$x(t) = t^2 + \int_0^t (t - s) \tanh x(s)\, ds, \ \ t \in [0, 1]. \tag{14}$$

Here, $q(t) = t^2 = x_0(t)$, $\sigma(t) = t$, $k(t, s) = t - s$ and $f(t, x) = \tanh x$ for all $t, s \in [0, 1]$ and $x \in \mathbb{R}$. Clearly, q is a continuous function and hence Lebesque integrable on J. The function k is continuous on $[0, 1] \times [0, 1]$ and hence satisfies integrability condition with $\gamma_k(s) \leq 1$ for all $s \in [0, 1]$. Next, the function f is continuous on $[0, 1] \times \mathbb{R}$ and so $L^1_{\mathbb{R}}$-Carathéodory with $h(s) = 1$ for all $s \in [0, , 1]$. Moreover, the map $x \mapsto \tanh x = f(t, x)$ is nondecreasing on $\mathbb{R}$ for all $t \in [0, 1]$. Finally, $f(t, , q(t)) = \tanh(t^2) \geq 0$ for all $t \in [0, 1]$. Thus, the functions q, q, k and f satisfy all the hypotheses (H_0), (H_1), (H_3), (H_4) and (H_5) of Theorem 6 with $r = 1$. Hence, the HFIE (14) has a integrable nonnegative local solution x^* in $B_1[x_0]$ and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$\begin{aligned} x_0(t) &= t^2, \ \ t \in [0, 1], \\ x_{n+1}(t) &= t^2 + \int_0^t (t - s) \tanh x_n(s)\, ds, \ \ t \in [0, 1], \end{aligned}$$

for $n = 0, 1, 2, \ldots$, is monotone nondeceasing and converges to x^*.

Example 2. Given $J = [0, 1] \subset \mathbb{R}$, consider the HFIE

$$x(t) = \frac{t+1}{2} + \frac{1}{2} \int_0^t (t - s) \tan^{-1} x(s)\, ds, \ \ t \in [0, 1]. \tag{15}$$

Here, $q(t) = \dfrac{t+1}{2} = x_0(t)$, $\sigma(t) = t$, $k(t, s) = t - s$ and $f(t, x) = \dfrac{1}{2} \tan^{-1} x$ for all $t \in [0, 1]$ and $x \in \mathbb{R}$. It can be shown as in example 1 that the functions q and k satisfy the hypotheses (H_0) and (H_1). Next, we show that the function f satisfies the hypothesis (H_2). Let $x, y \in \mathbb{R}$ be such that $x \geq y$. Then, since the map $x \mapsto \tan^{-1} x = f(t, x)$ is nondecreasing, we have

$$\begin{aligned} 0 \leq f(t, x) - f(t, y) &= \frac{1}{2} \tan^{-1} x - \frac{1}{2} \tan^{-1} y \\ &= \frac{1}{2} \big[\tan^{-1} x - \tan^{-1} y \big] \end{aligned}$$

$$\le \frac{1}{2} \cdot \frac{1}{1+\xi^2}(x-y) \qquad (x < \xi < y)$$
$$\le \frac{1}{2}(x-y),$$

for all $t \in [0,1]$ and $x, y \in \mathbb{R}$. This shows that hypothesis (H_2) is satisfied. Furthermore, $f(t, q(t)) = \tan^{-1}\left(\frac{t+1}{2}\right) \ge 0$ for all $t \in [0,1]$. Thus, the functions q, k and f satisfy all the hypotheses (H_0),(H_1), (H_2), (H_4), (H_5) and condition (13) of Theorem 7 with $r = 4$. Hence, the HIE (15) has a unique integrable positive local solution x^* in $B_4[x_0]$ and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$x_0(t) = \frac{t+1}{2}, \ \ t \in [0,1],$$
$$x_{n+1}(t) = \frac{t+1}{2} + \frac{1}{2}\int_0^t (t-s)\tan^{-1} x_n(s)\, ds, \ \ t \in [0,1],$$

for $n = 0, 1, 2, \ldots$, is monotone nondeceasing and converges to x^*.

Remark 6. *We note that the existence and uniqueness results, Therems 6 and 7 of this paper may be extended with appropriate modifications to the nonlinear hybrid Urysohn type functional integral equation,*

$$x(t) = q(t) + \int_0^{\sigma(t)} f(t, s, x(s))\, ds, \ \ t \in J, \tag{16}$$

with appropriate modifications. In this case the desired approximation results for existence and uniqueness theorems are obtained under the hypotheses similar to (H_2) through (H_5). The details of such criteria appear in Banas [2], *Emmanuel* [21] *and references therein.*

4. Comparison

We observe that the existence of solutions of the HFIE (1) can also be obtained by an application of topological Schauder fixed point principle under the hypothesis (H_0), (H_1) and (H_3) under certain restricted interval, but in that case we do not get any sequence of successive approximations that converges to the solution. Again, we can not apply analytical or geometric Banach contraction mapping principle to the problem (9) under the considered hypotheses (H_0), (H_1) and (H_2) in order to get the desired conclusion, because here the nonlinear function f does not satisfy the usual Lipschitz condition on the domain $J \times \mathbb{R}$. Similarly, since $L^1(J, \mathbb{R})$ is a complete lattice w.r.t. the partial $\preceq$, we can apply algebraic Tarski fixed point theorem [30] or its extension obtained in Dhage [6] to HFIE (1) under the hypotheses (H_0), (H_1), (H_3) and (H_4) for proving the existence of solution, but in that case also we do not get any sequence of successive approximations that converges to the solution. Therefore, all these arguments show that our hybrid fixed point principles, Theorems 4 and 5, have more advantages than other classical fixed point principles to get more information about the solution of nonlinear equations in the subject of nonlinear analysis. Finally, while concluding this paper, we mention that the integral equations (1) considered in this paper is very simple, however the method can be applied to other more complex nonlinear Volterra or Fredholm type integral equations involving integer or Riemann-Liouville type fractional order of integration. The research in this direction forms the further scope for the work and some of the results along this line will be reported elsewhere.

Acknowledgment

The authors are thankful to the referees for giving some useful suggestions for the improvement of his paper.

References

[1] Ardjouli A., Djoudi A., Approximating solutions of nonlinear hybrid Caputo fractional integrodifferential equations via Dhage iteration principle, Ural Math. Jour., 5 (1) (2019), 3-12.

[2] Banas J., Integrable solutions of Hammerstein and Urysohn integral equations, J. Austral. Math. Soc. (series A), 46 (1969), 61-68.

[3] Birkhoff G., Lattice Theory, Amer. Math. Soc. Coll. Publ. New York, 1967.

[4] Coddington E. A., An Introduction to Ordinary Differential Equations, Dover Publications Inc. New York, 1989.

[5] Deimling K., Nonlinear Functional Analysis, Springer Verlag, 1985.

[6] Dhage B. C., An extension of lattice fixed point theorem and its applications, Pure Appl. Math. Sci., 25 (1987), 37-42.

[7] Dhage B. C., Partially condensing mappings in partially ordered normed linear spaces and applications to functional integral equations, Tamkang J. Math., 45(4) (2014), 397-427.

[8] Dhage B. C., A coupled hybrid fixed point theorem for sum of two mixed monotone coupled operators in a partially ordered Banach space with applications, Tamkang J. Math., 50(1) (2019), 1-36.

[9] Dhage B. C., Coupled and mixed coupled hybrid fixed point principles in a partially ordered Banach algebra and PBVPs of nonlinear coupled quadratic differential equations, Differ. Equ. Appl., 11(1) (2019), 1-85.

[10] Dhage B. C., A Schauder type hybrid fixed point theorem in a partially ordered metric space with applications to nonlinear functional integral equations, Jñānābha, 52(2) (2022), 168-181.

[11] Dhage B. C., Nonlinear partial completely continuous operators in a partially ordered Banach space and nonlinear hyperbolic partial differential equations, Malaya J. Mat., 12(4) (2024), 330-338.

[12] Dhage B. C., Dhage S. B., Approximating solutions of nonlinear first order ordinary differential equations, GJMS Special issue for Recent Advances in Mathematical Sciences and Applications-13, GJMS, 2(2) (2014), 25-35.

[13] Dhage B. C., Dhage J. B., Dhage S. B., Approximating existence and uniqueness of solution to a nonlinear IVP of first order ordinary iterative differential equations, Nonlinear Studies, 29(1) (2022), 303-314.

[14] Dhage J. B., Dhage B. C., Approximating local solution of an IVP of nonlinear first order ordinary hybrid differential equations, Nonlinear Studies, 30(3) (2023), 721-732.

[15] Dhage J. B., Dhage B. C., Approximation, existence and uniqueness of the local integrable solution of nonlinear Volterra type hybrid integral equations, Mathematics Student, 93(3-4) (2024), 74-90.

[16] Dhage J. B., Dhage B. C., Some hybrid fixed point theorems for partial contraction mappings in an ordered Euclidean space $\mathbb{R}^n$, Jñānābha, 55(1) (2025), 188-196.

[17] Dhage J. B., Dhage S. B., Dhage B. C., Approximation results for PBVPs of nonlinear first order ordinary functional differential equations in a closed subset of the Banach space, Malaya Journal of Matematik, Vol. 11, No. S (2023), 197-207.

[18] Dhage J. B., Dhage S. B., Dhage B. C., An algorithmic approach to local solution of the nonlinear higher order ordinary hybrid differential equations, Jñānābha, 54(1) (2024), 29-40.

[19] Dhage B. C., Otrocol D., Dhage iteration method for approximating olutions of nonlinear differential equations with maxima, Fixed Point Theory, 19(2) (2018), 545-556.

[20] El-Sayed A. M. A., Al-Issa Sh. M., Existence of integrable solutions for integrodifferential inclusions of fractional order: coupled system approach, J. Nonlinear Sciences Appl., 13 (2020), 180-186.

[21] Emmanuel G., Integrable solutions of a functional-integral equations, J. Integral Equ. Appl., 4(1) (1992), 89-94.

[212 Granas A., Dugundji J., Fixed Point Theory, Springer 2003.

[23] Gua D., Lakshmikantham V., Nonlinear Problems in Abstract Cones, Academic Press, New York, London, 1988.

[24] Gupta V., Bora S. N., Nieto J. J., Dhage iteration principle for quadratic perturbations of fractional boundary value problem with finite delay, Math. Methods Appl. Sci., 42(12) (2019), 4244-4255.

[25] Heikkilä S., Lakshmikantham V., Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations, Marcel Dekker inc., New York, 1994.

[26] Krasnoselskii M. K., On continuity of superposition operator $Fx(t) = f(t, x(t))$, DOkl Akad Nauk SSSR, 77 (1951), 185-188.

[27] Ladde G. S., Lakshmikantham V., V, Vatsala A. S., Monotone Iterative Techniques for Nonlinear Differential Equations, Pitman, Boston, 1985.

[28] Mydlarczyk W., A system of Volterra integral equations with blowing up solutions, Colloq. Math., 146 (2017), 99-110.

[29] Raffoul Y., Classification of positive solutions of nonlinear systems of Volterra integral equations, Ann. Funct. Anal., 2 (2011), 34-41.

[30] Swartz C., Measure, Integration and Function Spaces, World Scientific Publishing Co. River Edge, 1999.

[31] Tarski A., A lattice theoretical fixed point theorem and its applications, Pacific J. Math., 5 (1965), 285-309.

[32] You J., Han Z., Analysis of fractional hybrid differential equations with impulses in partially ordered Banach algebras, Nonlinear Analysis: Modelling and Control, 26(6) (2021), 1071-1086.