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CHARACTERIZATION OF IDEMPOTENCY IN POWER-ASSOCIATIVE RINGS

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Abstract: In this paper, we extend Mosic's result for idempotency in associative rings to power-associative rings. We provide a necessary and sufficient condition for idempotency and give some examples.

Keywords and Phrases: Ring, power-associative, idempotent.

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1. Introduction

In this article we provide a necessary and sufficient condition for idempotency in power-associative rings, hence extending Mosic's result in [5]. Mosic gives the relation between idempotent and tripotent elements in an associative ring R, generalizing the result on matrices by Trenkler and Baksalary [8]. Namely, for any $x \in R$, where 2, 3 are invertible, x is idempotent if and only if x is tripotent and $1-x$ is tripotent or $1+x$ is invertible.

In [1], we pointed out that even though $\mathbb{O}/\mathbb{Z}_p{}^{1,2}$ is not associative, the result does hold in some cases. For example, consider the tripotent $x = 4+3e_1+e_2+4e_3$ in $\mathbb{O}/\mathbb{Z}_7$, which is also an idempotent. It is not hard to check this directly or using the conditions for idempotency in Theorem 3.1 in [1]. We noticed also that $1-x$ is tripotent and $1+x$ is invertible as $N(1+x)=\sqrt{2}\neq 0$. Similar things hold also for $x=4+e_1+3e_3+4e_5$, which is non-quaternionic (it is octonionic). So, given the above, we conjectured that Mosic's result may extend to some non-associative rings. As a matter of fact, it is true for general power-associative rings and hence to any associative, alternative and flexible ring.[3]

2. Extension to Power Associative Rings

In this section, we prove Mosic's result for general power-associative rings.

Theorem 1. *Let* $x \in R$ *where* R *is a power-associative ring and* $2, 3 \in R^{-1}$. *Then,* x *is idempotent if and only if* x *is tripotent and any of the following two conditions hold: (i)* $1-x$ *tripotent or (ii)* $1+x \in R^{-1}$.

Proof. Let x be idempotent, so $x^2 = x$. Then, $x^3 \overset{\text{pow.assoc.}}{=} x^2 x = xx = x^2 = x$. Hence, x is tripotent. In addition:

$$\begin{aligned}(1-x)^3 = (1-x)(1-x)(1-x) &\overset{\text{pow.assoc.}}{=} [(1-x)(1-x)](1-x)\\ &= (1-x-x+x^2)(1-x)\\ &= (1-x-x+x)(1-x)\\ &= (1-x)(1-x)\\ &= 1-x-x+x^2\\ &= 1-x-x+x\\ &= 1-x\end{aligned}$$

So, $1-x$ is tripotent. Also, $(1+x)(2-x) = 2-x+2x-x^2 = 2-x+2x-x = 2$. And since $2 \in R^{-1}$, then $1+x$ is right-invertible. Similarly, $(2-x)(1+x) = 2+2x-x-x^2 = 2+2x-x-x = 2$. And since $2 \in R^{-1}$, $1+x$ is also left-invertible. Therefore, $1+x$ is invertible. So, (i) and (ii) hold.

Conversely, let x be tripotent, i.e. $x^3 = x$, and suppose (i) or (ii) holds.

Case 1: x tripotent and (i) holds. We have $x^3 = x$ and $(1-x)^3 = 1-x$. Then:

$$\begin{aligned}(1-x)^3 = (1-x) &\implies (1-x)(1-x)(1-x) = 1-x\\ &\overset{\text{pow.assoc.}}{\implies} [(1-x)(1-x)](1-x) = 1-x\\ &\implies (1-x-x+x^2)(1-x) = 1-x\\ &\implies (1-2x+x^2)(1-x) = 1-x\\ &\overset{\text{pow.assoc.}}{\implies} 1-3x+3x^2-x^3 = 1-x\\ &\implies 1-3x+3x^2-x = 1-x\end{aligned}$$

$$\implies -3x + 3x^2 = 0$$
$$\implies 3x^2 = 3x$$

And since $3 \in R^{-1}$ we have that $x^2 = x$. Hence, x is idempotent.
Case 2: x tripotent and (ii) holds. We have $x^3 = x$ and $1 + x$ invertible. Then:

$$(1+x)x^2 \overset{\text{pow.assoc.}}{=} x^2 + x^3 = x^2 + x = x + x^2 \overset{\text{pow.assoc.}}{=} (1+x)x$$

And since $1 + x \in R^{-1}$ we have that $x^2 = x$. Hence, x is idempotent.
Both cases, (1) and (2), imply that x is idempotent. So, x is idempotent.

3. Examples

1. The finite ring $\mathbb{O}/\mathbb{Z}_7$ is alternative (non-associative) and therefore our theorem in Section 2 applies. We have already considered in Section 1 two elements in $\mathbb{O}/\mathbb{Z}_7$, namely $x = 4 + 3e_1 + e_2 + 4e_3$ and $x = 4 + e_1 + 3e_3 + 4e_5$, for which the theorem is true. These examples were mentioned in [1], and we noted already that the second example was non-quaternionic (it is octonionic). We provide one more octonionic element. Consider $x = 4 + e_1 + e_2 + e_3 + e_4 + e_5$ in $\mathbb{O}/\mathbb{Z}_7$. Then, x is idempotent, because $x^2 = 11 + 8e_1 + 8e_2 + 8e_3 + 8e_4 + 8e_5 \overset{\text{mod7}}{=} 4 + e_1 + e_2 + e_3 + e_4 + e_5$. It is also tripotent and $1 + x$ is invertible as $N(1 + x) = \sqrt{2} \neq 0$. Notice also that $1 - x$ is tripotent, as $1 - x = -3 - e_1 - e_2 - e_3 - e_4 - e_5 \overset{\text{mod7}}{=} 4 + 6e_1 + 6e_2 + 6e_3 + 6e_4 + 6e_5$ and $(1 - x)^3 = -2096 - 792e_1 - 792e_2 - 792e_3 - 792e_4 - 792e_5 \overset{\text{mod7}}{=} 4 + 6e_1 + 6e_2 + 6e_3 + 6e_4 + 6e_5 = 1 - x$.

2. It is important to note that all Cayley-Dickson algebras of dimension greater than 8 are non-alternative with zero divisors, also flexible (see [6]), and in particular power-associative. So, our theorem holds for all Cayley-Dickson algebras. For example, it holds for the flexible (non-alternative) Cayley-Dickson algebra of sedenions $\mathbb{S}$, which is constructed by applying the Cayley-Dickson process on octonions [3]. More specifically, consider $x = 2 - e_2 + e_{14}$ in the finite ring $\mathbb{S}/\mathbb{Z}_3$. Then, x is idempotent. It is also tripotent and $1 + x$ is invertible (its inverse is itself). Notice also that $1 - x$ is tripotent, as $1 - x = -1 + e_2 - e_{14} \overset{\text{mod3}}{=} 2 + e_2 + 2e_{14}$ and $(1 - x)^3 = -22 + 7e_2 + 14e_{14} \overset{\text{mod3}}{=} 2 + e_2 + 2e_{14} = 1 - x$.

3. Jordan Algebras are all power-associative and in general of non-Cayley-Dickson type. Hence our theorem applies. For example, consider the non-alternative Jordan algebra $\mathbb{J}_n(\mathbb{C})$ of self-adjoint nxn complex matrices, with the Jordan product $A \circ B = \frac{AB+BA}{2}$.[4] The element $X = \begin{pmatrix} 1/2 & i/2 \\ -i/2 & 1/2 \end{pmatrix}$ is idempotent, as $X^2 = X \circ X =$

$\frac{XX+XX}{2} = XX = \begin{pmatrix} 1/2 & i/2 \\ -i/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & i/2 \\ -i/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & i/2 \\ -i/2 & 1/2 \end{pmatrix} = X$. So, X is therefore tripotent and $1+X$ is invertible with inverse $(1+X)^{-1} = \begin{pmatrix} 3/4 & -i/4 \\ i/4 & 3/4 \end{pmatrix}$. Finally, notice that $I - X$ is also tripotent as $I - X = \bar{X}$ (the conjugate of X) and $(I-X)^3 = (\bar{X})^3 = \overline{(X^3)} = \overline{(X)} = \bar{X} = I - X$.

Notes

1. $\mathbb{O}$ is the octonions, one of the only four finite dimensional normed division algebras. $\mathbb{O}/\mathbb{Z}_p$ is the finite ring of octonions with coefficients from $\mathbb{Z}_p$.
2. The multiplication in $\mathbb{O}$ is given by the Fano Plane (Figure 1) or the Multiplication Table (Figure 2) below (see [3]), which is the multiplication table for the sedenions $\mathbb{S}$ and includes the multiplication tables for the division algebras $\mathbb{R}, \mathbb{H}$ and $\mathbb{O}$. (set $e_i = i$ for simplicity, i=1,2,...,15).

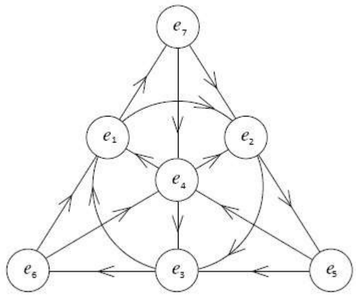

Figure 1: Fano Plane

	*	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
C	1	1	-0	3	-2	5	-4	-7	6	9	-8	-11	10	-13	12	15	-14
	2	2	-3	-0	1	6	7	-4	-5	10	11	-8	-9	-14	-15	12	13
H	3	3	2	-1	-0	7	-6	5	-4	11	-10	9	-8	-15	14	-13	12
	4	4	-5	-6	-7	-0	1	2	3	12	13	14	15	-8	-9	-10	-11
	5	5	4	-7	6	-1	-0	-3	2	13	-12	15	-14	9	-8	11	-10
	6	6	7	4	-5	-2	3	-0	-1	14	-15	-12	13	10	-11	-8	9
O	7	7	-6	5	4	-3	-2	1	-0	15	14	-13	-12	11	10	-9	-8
	8	8	-9	-10	-11	-12	-13	-14	-15	-0	1	2	3	4	5	6	7
	9	9	8	-11	10	-13	12	15	-14	-1	-0	-3	2	-5	4	7	-6
	10	10	11	8	-9	-14	-15	12	13	-2	3	-0	-1	-6	-7	4	5
	11	11	-10	9	8	-15	14	-13	12	-3	-2	1	-0	-7	6	-5	4
	12	12	13	14	15	8	-9	-10	-11	-4	5	6	7	-0	-1	-2	-3
	13	13	-12	15	-14	9	8	11	-10	-5	-4	7	-6	1	-0	3	-2
	14	14	-15	-12	13	10	-11	8	9	-6	-7	-4	5	2	-3	-0	1
	15	15	14	-13	-12	11	10	-9	8	-7	6	-5	-4	3	2	-1	-0

Figure 2: Multiplication Table

3. A magma is a set equipped with a binary operation $\star$ that is closed under $\star$. A magma is said to be power-associative if the subalgebra generated by any of its elements is associative. Concretely, this means that if an element x operates via $\star$ with itself several times, it does not matter in which order the operations are carried out. So, for instance, $x \star (x \star (x \star x)) = (x \star (x \star x)) \star x = (x \star x) \star (x \star x)$. It is immediate then, by their defining relations, that the associative, alternative and flexible algebras are also power-associative algebras
4. $\mathbb{J}_n(\mathbb{C})$ is isomorphic to the spin algebra $\mathbb{R}^3 \oplus \mathbb{R}$ and hence is of non Cayley Dickson type. For more see [4].

References

[1] Aristidou M., Brown P. R. and Chailos G., Idempotent and Nilpotent Elements in Octonion Rings over $\mathbb{Z}_p$, Stud. Univ. Babeş-Bolyai Math, 69(1) (2024), 3-14.

[2] Baez J. C., The Octonions, Bull. Amer. Math. Soc., 39 (2002), 145-205.

[3] Cawagas R. E., On the Structure and Zero Divisors of the Cayley-Dickson Sedenion Algebra, Discus. Math., Gen. Algebra Appl., 24 (2004), 251-265.

[4] Jordan P., Neumann J. Von, Wigner E., On an algebraic generalization of the quantum mechanical formalism, Annals of Mathematics, 35(1), (1934), 29–64.

[5] Mosic D., Characterizations of k-potent Elements in Rings, Ann. Mat. Pure Appl., 194(4) (2015), 1157-1168.

[6] Schafer R., On Algebras formed by Cayley-Dickson Process, Amer. J. Math, (1954), 435-446.

[7] Schafer R., An Introduction to Nonassociative Algebras, Academic Press, 1996.

[8] Trenkler G. and Baksalary O. M., On k-potent Matrices, Electron J. Linear Algebra, 26 (2013), 446-470.

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