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# ON THE VALUE DISTRIBUTION OF SOME DIFFERENTIAL POLYNOMIALS WHICH INVOLVE TWO DISTINCT TRANSCENDENTAL MEROMORPHIC FUNCTIONS

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**Abstract:** In the present note we deal with the value distribution of those differential polynomials which involve two distinct transcendental meromorphic functions and obtain analogous results of G. P. Barker and A. P. Singh [1], C. C. Yang [6], W. Doeringer [3], X. Z. Xiao and Y. Z. He [5], Hong-Xun Yi [7], S. S. Bhoosnurmath and K. S. L. N. Prasad [2] for these kind of differential polynomials.

**Keywords and Phrases:** Differential polynomials, monomials, transcendental meromorphic functions, small functions, value distribution.

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## 1. Introduction, Definitions and Notations

Let $f$ be a non-constant meromorphic function in the complex plane and $m(r, f)$, $N(r, f)$, $T(r, f)$ have their usual meanings in the Nevanlinna Theory [4]. Let $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ except possibly a set of finite linear measure. Let $a(z)$ be a meromorphic function in the plane satisfying $T(r, a(z)) = S(r, f)$ as $r \to \infty$.

**Definition 1.1.** *A monomial in $f$ is an expression of the form*

$$M[f] = (f(z))^{l_0}(f^{(1)}(z))^{l_1}...(f^{(k)}(z))^{l_k}$$

*where* $l_0$, $l_1$, ... , $l_k$ *are non-negative integers.*

Here we denote

$$\gamma_M = (l_0 + l_1 + ... + l_k)$$

to be the degree of the monomial and

$$\Gamma_M = l_0 + 2l_1 + 3l_2 + ... + (k+1)l_k$$

to be the weight of the monomial.
Now if $M_1[f]$, $M_2[f]$, ... , $M_n[f]$ denote monomials in $f$ then

$$Q[f] = a_1 M_1[f] + a_2 M_2[f] + ... + a_n M_n[f]$$

with $a_i \not\equiv 0$ $(i = 1, 2, ..., n)$ is called a differential polynomial in $f$ of degree $\gamma_Q = max\{\gamma_{M_i} : 1 \leq i \leq n\}$ and weight $\Gamma_Q = max\{\Gamma_{M_i} : 1 \leq i \leq n\}$.
If $\gamma_{M_1} = \gamma_{M_2} = ... = \gamma_{M_n} = \gamma_Q$ then $Q[f]$ will be called a homogeneous differential polynomial of degree $\gamma_Q$.
Let $f$ and $g$ be two non-constant meromorphic functions in the complex plane and $S(r, f)$, $S(r, g)$ denote any quantities satisfying $S(r, f) = o(T(r, f))$ and $S(r, g) = o(T(r, g))$ respectively as $r \to \infty$ except possibly a set of finite linear measure. Let $a(z)$ be a meromorphic function in the plane satisfying $T(r, a(z)) = S(r, f)$ and $T(r, a(z)) = S(r, g)$ as $r \to \infty$.
Now we introduce the analogous definitions for two non-constant meromorphic functions $f$ and $g$.

**Definition 1.2.** *A monomial in* $f$ *and* $g$ *is an expression of the form*

$$M[f, g] = (f(z))^{l_0}(f^{(1)}(z))^{l_1}...(f^{(k)}(z))^{l_k}(g(z))^{m_0}(g^{(1)}(z))^{m_1}...(g^{(j)}(z))^{m_j}$$

*where* $l_0$, $l_1$, ... , $l_k$, $m_0$, $m_1$, ... , $m_j$ *are non-negative integers.*

Here we denote

$$\gamma_M = (l_0 + l_1 + ... + l_k) + (m_0 + m_1 + ... + m_j)$$

to be the degree of the monomial and

$$\Gamma_M = \{l_0 + 2l_1 + 3l_2 + ... + (k+1)l_k\} + \{m_0 + 2m_1 + 3m_2 + ... + (j+1)m_j\}$$

to be the weight of the monomial.
Now if $M_1[f, g]$, $M_2[f, g]$, ... , $M_n[f, g]$ denote monomials in $f$ and $g$ then

$$Q[f, g] = a_1 M_1[f, g] + a_2 M_2[f, g] + ... + a_n M_n[f, g]$$

with $a_i \not\equiv 0$ $(i = 1, 2, ..., n)$ is called a differential polynomial in $f$ and $g$ of degree $\gamma_Q = max\{\gamma_{M_i} : 1 \le i \le n\}$ and weight $\Gamma_Q = max\{\Gamma_{M_i} : 1 \le i \le n\}$.
Evidently, if $\gamma_{M_1} = \gamma_{M_2} = ... = \gamma_{M_n} = \gamma_Q$ then $Q[f, g]$ will be called a homogeneous differential polynomial of degree $\gamma_Q$.

## 2. Preliminary Results

In [6], C. C. Yang proved the following theorem.

**Theorem 2.1.** *Let $f(z)$ be a transcendental meromorphic function with $N(r, f) = S(r, f)$. If*

$$P[f] = (f)^n + a_1\pi_{n-1}[f] + a_2\pi_{n-2}[f] + ... + a_{n-1}\pi_1[f] + a_n \tag{2.1}$$

*where each $\pi_i[f]$ is a homogeneous differential polynomial in $f$ of degree $i$, then*

$$T(r, P[f]) = nT(r, f) + S(r, f).$$

Using the result of Theorem 2.1, S. S. Bhoosnurmath and K. S. L. N. Prasad [2] proved the following theorem.

**Theorem 2.2.** *Let $f$ be a transcendental meromorphic function in the complex plane and $Q_1[f]$, $Q_2[f]$ be differential polynomials in $f$ satisfying $Q_1[f] \not\equiv 0$, $Q_2[f] \not\equiv 0$ and $P[f]$ is defined by* (2.1). *Now if*

$$F = P[f]Q_1[f] + Q_2[f]$$

*then*

$$(n - \gamma_{Q_2})T(r, f) \le \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{P[f]}) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\bar{N}(r, f) + S(r, f).$$

The proof of Theorem 2.2 is based on the following four lemmas.

**Lemma 2.1.** [7] *If $Q[f]$ is a differential polynomial in $f$ with arbitrary meromorphic coefficients $q_i$, $1 \le i \le n$, then*

$$m(r, Q[f]) \le \gamma_Q m(r, f) + \sum_{i=1}^{n} m(r, q_i) + S(r, f).$$

**Lemma 2.2.** [7] *Let $Q^*[f]$ and $Q[f]$ denote differential polynomials in $f$ with arbitrary meromorphic coefficients ${q_1}^*, {q_2}^*, ..., {q_n}^*$ and $q_1, q_2, ..., q_k$ respectively. Suppose that $P[f]$ is given by (2.1). If $P[f]Q^*[f] = Q[f]$ and $\gamma_Q \le n$, then*

$$m(r, Q^*[f]) \le \sum_{i=1}^{n} m(r, {q_i}^*) + \sum_{i=1}^{k} m(r, q_i) + S(r, f).$$

**Lemma 2.3.** [7] *Suppose that $M[f]$ is a monomial in $f$. If $f$ has a pole at $z = z_0$ of order $m$, then $z_0$ is a pole of $M[f]$ of order $(m-1)\gamma_M + \Gamma_M$.*

**Lemma 2.4.** [7] *Suppose that $Q[f]$ is a differential polynomial in $f$. Let $z_0$ be a pole of $f$ of order $m$ and not a zero or a pole of coefficients of $Q[f]$. Then $z_0$ is a pole of $Q[f]$ of order at most $m\gamma_Q + (\Gamma_Q - \gamma_Q)$.*

## 3. Main Results

In this section we present our main results of the paper.

**Theorem 3.1.** *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with $N(r, f) = S(r, f)$ and $N(r, g) = S(r, g)$. Also let*

$$\begin{aligned} P[f,g] = f^l g^m &+ a_1\pi_{(l-1,m)}[f,g] + b_1\pi_{(l,m-1)}[f,g] + a_2\pi_{(l-2,m)}[f,g] \\ &+ b_2\pi_{(l,m-2)}[f,g] + ... + a_l\pi_{(0,m)}[f,g] + b_m\pi_{(l,0)}[f,g] \end{aligned} \tag{3.1}$$

*where each $\pi_{(l,m)}[f,g]$ is a homogeneous differential polynomial in $f$ and $g$ of degree $(l+m)$ having sum of finite terms*

$$(f(z))^{l_0}(f^{(1)}(z))^{l_1}...(f^{(k)}(z))^{l_k}(g(z))^{m_0}(g^{(1)}(z))^{m_1}...(g^{(j)}(z))^{m_j}$$

*such that $(l_0 + l_1 + ... + l_k) = l$ and $(m_0 + m_1 + ... + m_j) = m$; $a_1, a_2, ..., a_l$ being small functions of $f$ and $b_1, b_2, ..., b_m$ being small functions of $g$. Now if $|f(re^{i\theta})| = |g(re^{i\theta})| \geq 1$ on the circle $|z| = r$, then*

$$\begin{aligned} \frac{1}{2}[lT(r,f) + mT(r,g)] &\leq T(r, P[f,g]) + S(r,f) + S(r,g) \\ &\leq 2[lT(r,f) + mT(r,g)]. \end{aligned} \tag{3.2}$$

**Proof.** From (3.1) we have

$$\begin{aligned} P[f,g] &= f^l g^m\{1 + \frac{a_1\pi_{(l-1,m)}[f,g]}{f^l g^m} + \frac{b_1\pi_{(l,m-1)}[f,g]}{f^l g^m} + \frac{a_2\pi_{(l-2,m)}[f,g]}{f^l g^m} \\ &\quad + \frac{b_2\pi_{(l,m-2)}[f,g]}{f^l g^m} + ... + \frac{a_l\pi_{(0,m)}[f,g]}{f^l g^m} + \frac{b_m\pi_{(l,0)}[f,g]}{f^l g^m}\} \\ &= f^l g^m\{1 + \frac{A_1}{f} + \frac{B_1}{g} + \frac{A_2}{f^2} + \frac{B_2}{g^2} + ... + \frac{A_l}{f^l} + \frac{B_m}{g^m}\} \quad (say) \end{aligned} \tag{3.3}$$

where

$$A_i = \frac{a_i\pi_{(l-i,m)}[f,g]}{(f)^{l-i}(g)^m}, \quad for \;\; i = 0, 1, 2, ..., l$$

and

$$B_i = \frac{b_i \pi_{(l,m-i)}[f,g]}{(f)^l (g)^{m-i}}, \quad for \ \ i = 0, 1, 2, ..., m.$$

Now since $\pi_{(l,m)}[f,g]$ is a homogeneous differential polynomial in $f$ and $g$ of degree $(l+m)$, precisely, a finite sum of terms

$$(f(z))^{l_0}(f^{(1)}(z))^{l_1}...(f^{(k)}(z))^{l_k}(g(z))^{m_0}(g^{(1)}(z))^{m_1}...(g^{(j)}(z))^{m_j}$$

such that $(l_0 + l_1 + ... + l_k) = l$ and $(m_0 + m_1 + ... + m_j) = m$, we have

$$\begin{aligned}
m(r, \frac{\pi_{(l,m)}[f,g]}{(f)^l(g)^m}) &= m(r, \frac{\sum (f)^{l_0}(f^{(1)})^{l_1}...(f^{(k)})^{l_k}(g)^{m_0}(g^{(1)})^{m_1}...(g^{(j)})^{m_j}}{(f)^l(g)^m}) \\
&= m(r, \sum (\frac{f^{(1)}}{f})^{l_1}(\frac{f^{(2)}}{f})^{l_2}...(\frac{f^{(k)}}{f})^{l_k}(\frac{g^{(1)}}{g})^{m_1}(\frac{g^{(2)}}{g})^{m_2}...(\frac{g^{(j)}}{g})^{m_j}) \\
&\leq \sum m(r, (\frac{f^{(1)}}{f})^{l_1}(\frac{f^{(2)}}{f})^{l_2}...(\frac{f^{(k)}}{f})^{l_k}(\frac{g^{(1)}}{g})^{m_1}(\frac{g^{(2)}}{g})^{m_2}...(\frac{g^{(j)}}{g})^{m_j}) + O(1) \\
&\leq \sum [l_1 m(r, \frac{f^{(1)}}{f}) + l_2 m(r, \frac{f^{(2)}}{f}) + ... + l_k m(r, \frac{f^{(k)}}{f}) + m_1 m(r, \frac{g^{(1)}}{g}) \\
&\qquad + m_2 m(r, \frac{g^{(2)}}{g}) + ... + m_j m(r, \frac{g^{(j)}}{g})] + O(1) \\
&= S(r,f) + S(r,g), \qquad using\ Milloux's\ Theorem.
\end{aligned}$$

Which implies

$$m(r, \frac{A_0}{a_0}) = S(r,f) + S(r,g) \tag{3.4}$$

and similarly

$$m(r, \frac{B_0}{b_0}) = S(r,f) + S(r,g). \tag{3.5}$$

Hence using (3.4) and the fact that $a_i$'s are small functions of f, we have

$$\begin{aligned}
m(r, A_0) &\leq m(r, \frac{A_0}{a_0}) + m(r, a_0) \\
&= S(r,f) + S(r,g)
\end{aligned}$$

and likewise, using (3.5) and the fact that $b_i$'s are small functions of g, we have

$$\begin{aligned}
m(r, B_0) &\leq m(r, \frac{B_0}{b_0}) + m(r, b_0) \\
&= S(r,f) + S(r,g).
\end{aligned}$$

Similarly, for all $i = 1, 2, ..., l$ it follows that

$$m(r, A_i) = S(r, f) + S(r, g) \tag{3.6}$$

and also for all $i = 1, 2, ..., m$ it follows that

$$m(r, B_i) = S(r, f) + S(r, g). \tag{3.7}$$

Now on the circle $|z| = r$, let

$$A(re^{i\theta}) = max\{|A_1(re^{i\theta})|, |A_2(re^{i\theta})|^{\frac{1}{2}}, ..., |A_l(re^{i\theta})|^{\frac{1}{l}}\}$$

and

$$B(re^{i\theta}) = max\{|B_1(re^{i\theta})|, |B_2(re^{i\theta})|^{\frac{1}{2}}, ..., |B_m(re^{i\theta})|^{\frac{1}{m}}\}.$$

Then from (3.6), we have

$$m(r, A(z)) = S(r, f) + S(r, g) \tag{3.8}$$

and from (3.7), we have

$$m(r, B(z)) = S(r, f) + S(r, g). \tag{3.9}$$

Again let

$$E_1 = \{\theta \in [0, 2\pi] : |f(re^{i\theta})| > 4A(re^{i\theta})\}$$

and

$$E_2 = \{\theta \in [0, 2\pi] : |g(re^{i\theta})| > 4B(re^{i\theta})\}.$$

Then on $E_1 \cap E_2$, we have from (3.3)

$$\begin{aligned}
|P[f,g]| &= |f|^l |g|^m |1 + \frac{A_1}{f} + \frac{A_2}{f^2} + ... + \frac{A_l}{f^l} + \frac{B_1}{g} + \frac{B_2}{g^2} + ... + \frac{B_m}{g^m}| \\
&\geq |f|^l |g|^m [1 - |\frac{A_1}{f}| - |\frac{A_2}{f^2}| - ... - |\frac{A_l}{f^l}| - |\frac{B_1}{g}| - |\frac{B_2}{g^2}| - ... - |\frac{B_m}{g^m}|] \\
&\geq |f|^l |g|^m [1 - |\frac{A}{f}| - |\frac{A}{f}|^2 - ... - |\frac{A}{f}|^l - |\frac{B}{g}| - |\frac{B}{g}|^2 - ... - |\frac{B}{g}|^m], \\
&\qquad \textit{since on the circle } |z| = r, A \geq |A_i|^{\frac{1}{i}} \textit{ for all } i = 1, 2, ..., l \\
&\qquad \textit{and } B \geq |B_i|^{\frac{1}{i}} \textit{ for all } i = 1, 2, ..., m
\end{aligned}$$

$$\geq |f|^l|g|^m[1 - \frac{1}{4} - (\frac{1}{4})^2 - ... - (\frac{1}{4})^l - \frac{1}{4} - (\frac{1}{4})^2 - ... - (\frac{1}{4})^m],$$

$$\textit{since on } E_1 \cap E_2, \quad |\frac{A}{f}| < \frac{1}{4} \textit{ and } |\frac{B}{g}| < \frac{1}{4}$$

$$= |f|^l|g|^m[1 - \{\frac{1}{4} + (\frac{1}{4})^2 + ... + (\frac{1}{4})^l\} - \{\frac{1}{4} + (\frac{1}{4})^2 + ... + (\frac{1}{4})^m\}]$$

$$\geq |f|^l|g|^m[1 - \{\frac{\frac{1}{4}}{1 - \frac{1}{4}}\} - \{\frac{\frac{1}{4}}{1 - \frac{1}{4}}\}]$$

$$= |f|^l|g|^m[1 - \frac{1}{3} - \frac{1}{3}]$$

$$= \frac{1}{3}|f|^l|g|^m.$$

Hence on $E_1 \cap E_2$ we have

$$3|P[f,g]| \geq |f|^l|g|^m$$

$$\textit{i.e.,} \quad \log^+ 3|P[f,g]| \geq \log^+ |f|^l|g|^m.$$

So,

$$l \log^+ |f| \leq \log 3 + \log^+ |P[f,g]| \tag{3.10}$$

and

$$m \log^+ |g| \leq \log 3 + \log^+ |P[f,g]|. \tag{3.11}$$

Therefore using (3.10) and by our hypothesis that on the circle $|z| = r$, $|f(re^{i\theta})| = |g(re^{i\theta})|$, on ${E_1}^c$, $|f(re^{i\theta})| \leq 4A(re^{i\theta})$ and on $E_1 \cap {E_2}^c$, $|g(re^{i\theta})| \leq 4B(re^{i\theta})$, we have

$$\begin{aligned} l \times m(r,f) &= l \times \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|d\theta \\ &= \frac{1}{2\pi}\int_{E_1 \cap E_2} l \log^+ |f(re^{i\theta})|d\theta + \frac{l}{2\pi}\int_{E_1 \cap {E_2}^c} \log^+ |f(re^{i\theta})|d\theta \\ &\qquad + \frac{l}{2\pi}\int_{{E_1}^c} \log^+ |f(re^{i\theta})|d\theta \\ &\leq \frac{1}{2\pi}\int_{E_1 \cap E_2} (\log 3 + \log^+ |P[f,g]|)d\theta + \frac{l}{2\pi}\int_{E_1 \cap {E_2}^c} \log^+ |g(re^{i\theta})|d\theta \\ &\qquad + \frac{l}{2\pi}\int_{{E_1}^c} \log^+ 4A \ d\theta \\ &\leq \frac{1}{2\pi}\int_{E_1 \cap E_2} \log 3 d\theta + \frac{1}{2\pi}\int_{E_1 \cap E_2} \log^+ |P[f,g]|d\theta \end{aligned}$$

$$
\begin{aligned}
&+ \frac{l}{2\pi}\int_{E_1\cap {E_2}^c} \log^+ 4B\ d\theta + \frac{l}{2\pi}\int_{{E_1}^c} \log^+ 4A\ d\theta \\
\le\ & \frac{1}{2\pi}\int_0^{2\pi} \log 3 d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log^+ |P[f,g]| d\theta + \frac{l}{2\pi}\int_0^{2\pi} \log^+ 4B\ d\theta \\
&+ \frac{l}{2\pi}\int_0^{2\pi} \log^+ 4A\ d\theta \\
=\ & \log 3 + \frac{1}{2\pi}\int_0^{2\pi} \log^+ |P[f,g]| d\theta + \frac{l}{\pi}\int_0^{2\pi} \log^+ 4 d\theta \\
&+ \frac{l}{2\pi}\int_0^{2\pi} \log^+ B\ d\theta + \frac{l}{2\pi}\int_0^{2\pi} \log^+ A\ d\theta \\
=\ & \log 3 + m(r, P[f,g]) + 2l \log 4 + l \times m(r,B) + l \times m(r,A) \\
=\ & m(r, P[f,g]) + S(r,f) + S(r,g), \qquad \textit{using } (3.8) \textit{ and } (3.9).
\end{aligned}
$$

Adding $l \times N(r,f)$ on both sides and recalling that $N(r,f) = S(r,f)$ we get

$$
\begin{aligned}
l \times T(r,f) &\le m(r, P[f,g]) + S(r,f) + S(r,g) \\
&\le T(r, P[f,g]) + S(r,f) + S(r,g). \qquad (3.12)
\end{aligned}
$$

Similarly, using (3.11) and by our hypothesis that on the circle $|z| = r$, $|f(re^{i\theta})| = |g(re^{i\theta})|$, on ${E_2}^c$, $|g(re^{i\theta})| \le 4B(re^{i\theta})$ and on ${E_1}^c \cap E_2$, $|f(re^{i\theta})| \le 4A(re^{i\theta})$, we have

$$
\begin{aligned}
m \times m(r,g) &= m \times \frac{1}{2\pi}\int_0^{2\pi} \log^+ |g(re^{i\theta})| d\theta \\
&= \frac{1}{2\pi}\int_{E_1\cap E_2} m \log^+ |g(re^{i\theta})| d\theta + \frac{m}{2\pi}\int_{{E_1}^c\cap E_2} \log^+ |g(re^{i\theta})| d\theta \\
&\qquad + \frac{m}{2\pi}\int_{{E_2}^c} \log^+ |g(re^{i\theta})| d\theta \\
&\le \frac{1}{2\pi}\int_{E_1\cap E_2} (\log 3 + \log^+ |P[f,g]|) d\theta + \frac{m}{2\pi}\int_{{E_1}^c\cap E_2} \log^+ |f(re^{i\theta})| d\theta \\
&\qquad + \frac{m}{2\pi}\int_{{E_2}^c} \log^+ 4B\ d\theta \\
&\le \frac{1}{2\pi}\int_{E_1\cap E_2} \log 3 d\theta + \frac{1}{2\pi}\int_{E_1\cap E_2} \log^+ |P[f,g]| d\theta \\
&\qquad + \frac{m}{2\pi}\int_{{E_1}^c\cap E_2} \log^+ 4A d\theta + \frac{m}{2\pi}\int_{{E_2}^c} \log^+ 4B d\theta
\end{aligned}
$$

$$\le \frac{1}{2\pi}\int_0^{2\pi} \log 3 d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log^+ |P[f,g]| d\theta + \frac{m}{2\pi}\int_0^{2\pi} \log^+ 4A\ d\theta$$
$$+ \frac{m}{2\pi}\int_0^{2\pi} \log^+ 4B\ d\theta$$
$$= \log 3 + \frac{1}{2\pi}\int_0^{2\pi} \log^+ |P[f,g]| d\theta + \frac{m}{\pi}\int_0^{2\pi} \log^+ 4 d\theta + \frac{m}{2\pi}\int_0^{2\pi} \log^+ A\ d\theta$$
$$+ \frac{m}{2\pi}\int_0^{2\pi} \log^+ B\ d\theta$$
$$= \log 3 + m(r, P[f,g]) + 2m \log 4 + m \times m(r,A) + m \times m(r,B)$$
$$= m(r, P[f,g]) + S(r,f) + S(r,g), \qquad \textit{using } (3.8) \textit{ and } (3.9).$$

Adding $m \times N(r,g)$ on both sides and recalling that $N(r,g) = S(r,g)$ we get

$$\begin{aligned} m \times T(r,g) &\le m(r, P[f,g]) + S(r,f) + S(r,g) \\ &\le T(r, P[f,g]) + S(r,f) + S(r,g). \end{aligned} \tag{3.13}$$

Now adding (3.12) and (3.13), we get

$$l \times T(r,f) + m \times T(r,g) \le 2T(r, P[f,g]) + S(r,f) + S(r,g)$$
$$i.e., \quad \frac{1}{2}[l \times T(r,f) + m \times T(r,g)] \le T(r, P[f,g]) + S(r,f) + S(r,g). \tag{3.14}$$

Next from (3.3) we have

$$\begin{aligned} m(r, P[f,g]) &= m(r, f^l g^m + A_1 f^{l-1} g^m + B_1 f^l g^{m-1} \\ &\quad + A_2 f^{l-2} g^m + B_2 f^l g^{m-2} + ... + A_l g^m + B_m f^l) \\ &\le m(r, f^l g^m + A_1 f^{l-1} g^m + A_2 f^{l-2} g^m + ... + A_l g^m) \\ &\quad + m(r, B_1 f^l g^{m-1} + B_2 f^l g^{m-2} + ... + B_m f^l) + \log 2 \\ &= m(r, g^m\{f^l + A_1 f^{l-1} + A_2 f^{l-2} + ... + A_l\}) \\ &\quad + m(r, f^l\{B_1 g^{m-1} + B_2 g^{m-2} + ... + B_m\}) + O(1) \\ &\le m(r, g^m) + m(r, f^l + A_1 f^{l-1} + A_2 f^{l-2} + ... + A_l) \\ &\quad + m(r, f^l) + m(r, B_1 g^{m-1} + B_2 g^{m-2} + ... + B_m) + O(1) \\ &\le m \times m(r,g) + m(r, f^l + A_1 f^{l-1} + A_2 f^{l-2} + ... + A_{l-1} f) \\ &\quad + m(r, A_l) + l \times m(r,f) + m(r, B_1 g^{m-1} + B_2 g^{m-2} + ... \\ &\quad + B_{m-1} g) + m(r, B_m) + O(1) \\ &\le m \times m(r,g) + m(r, f\{f^{l-1} + A_1 f^{l-2} + ... + A_{l-2} f + A_{l-1}\}) \\ &\quad + l \times m(r,f) + m(r, g\{B_1 g^{m-2} + B_2 g^{m-3} + ... + B_{m-2} g \end{aligned}$$

$$
\begin{aligned}
&+ B_{m-1}\}) + S(r,f) + S(r,g), \quad \textit{using } (3.6) \textit{ and } (3.7)\\
\leq\ & (l+1) \times m(r,f) + (m+1) \times m(r,g) + m(r, f^{l-1} + A_1 f^{l-2} + \ldots\\
&+ A_{l-2} f + A_{l-1}) + m(r, B_1 g^{m-2} + B_2 g^{m-3} + \ldots + B_{m-2} g\\
&+ B_{m-1}) + S(r,f) + S(r,g)\\
\leq\ & (l+1) \times m(r,f) + (m+1) \times m(r,g) + m(r, f\{f^{l-2} + A_1 f^{l-3} + \ldots\\
&+ A_{l-3} f + A_{l-2}\}) + m(r, g\{B_1 g^{m-3} + B_2 g^{m-4} + \ldots + B_{m-3} g\\
&+ B_{m-2}\}) + S(r,f) + S(r,g)\\
\leq\ & (l+2) \times m(r,f) + (m+2) \times m(r,g) + m(r, f^{l-2} + A_1 f^{l-3} + \ldots\\
&+ A_{l-3} f + A_{l-2}) + m(r, B_1 g^{m-3} + B_2 g^{m-4} + \ldots + B_{m-3} g\\
&+ B_{m-2}) + S(r,f) + S(r,g).
\end{aligned}
$$

Proceeding similarly, we ultimately get

$$
\begin{aligned}
m(r, P[f,g]) &\leq 2l \times m(r,f) + (2m-1) \times m(r,g) + S(r,f) + S(r,g)\\
&\leq 2l \times m(r,f) + 2m \times m(r,g) + S(r,f) + S(r,g). \qquad (3.15)
\end{aligned}
$$

Finally using (3.15) and by our hypotheses that $N(r,f) = S(r,f)$ and $N(r,g) = S(r,g)$, we get

$$
\begin{aligned}
T(r, P[f,g]) &= m(r, P[f,g]) + N(r, P[f,g])\\
&\leq 2l \times m(r,f) + 2m \times m(r,g) + S(r,f) + S(r,g)\\
&\leq 2[l \times T(r,f) + m \times T(r,g)] + S(r,f) + S(r,g). \qquad (3.16)
\end{aligned}
$$

Combining (3.14) and (3.16) we obtain the result of the theorem.
Here the condition that $|f(re^{i\theta})| = |g(re^{i\theta})| \geq 1$ on the circle $|z| = r$ can not be dropped. To show the necessity of this condition we can consider the following example.

**Example 3.1.** Let $f(z) = e^z(z-2)$ and $g(z) = e^{-z}$ where $P[f,g] = fg$.
Here $l = 1$, $m = 1$, $T(r,f) = \frac{r}{\pi} + O(\log r)$, $T(r,g) = \frac{r}{\pi}$ and $T(r, P[f,g]) = \log r + O(1)$ for $r \geq 2$.
Hence the inequality $\frac{1}{2}[lT(r,f) + mT(r,g)] \leq T(r, P[f,g]) + S(r,f) + S(r,g)$ does not hold here for large $r$. This happens because for large $r$, $|g(re^{i\theta})| \ngeq 1$ on the circle $|z| = r$.
Also Theorem 3.1 fails to hold if $f$ and $g$ are not transcendental. To show this we consider the following example.

**Example 3.2.** Let $f(z) = (z-2)^2$ and $g(z) = \frac{1}{z-2}$ where $P[f,g] = fg$.

Here $l = 1$, $m = 1$, $T(r, f) = 2\log r + O(1)$, $T(r, g) = \log r + O(1)$ and $T(r, P[f, g]) = \log r + O(1)$.
Hence the inequality $\frac{1}{2}[lT(r, f) + mT(r, g)] \leq T(r, P[f, g]) + S(r, f) + S(r, g)$ does not hold here. This happens because $f$ and $g$ are not transcendental here although $|f(re^{i\theta})| \geq 1$ but $|g(re^{i\theta})| \ngeq 1$ on the circle $|z| = r$ when $r$ is large.
Now we prove the following lemmas which will be needed to prove Theorem 3.2.

**Lemma 3.1.** *If $Q[f, g]$ is a differential polynomial in $f$ and $g$ with arbitrary meromorphic coefficients $q_i$, $1 \leq i \leq n$, then*

$$m(r, Q[f, g]) \leq 2n\gamma_Q[m(r, f) + m(r, g)] + \sum_{i=1}^{n} m(r, q_i) + S(r, f) + S(r, g).$$

**Proof.** Let

$$Q[f, g] = q_1M_1[f, g] + q_2M_2[f, g] + ... + q_nM_n[f, g] \tag{3.17}$$

where each

$$M_i[f, g] = (f(z))^{l_{i0}}(f^{(1)}(z))^{l_{i1}}...(f^{(k)}(z))^{l_{ik}}(g(z))^{m_{i0}}(g^{(1)}(z))^{m_{i1}}...(g^{(j)}(z))^{m_{ij}}$$

is a monomial in $f$ and $g$ of degree

$$\gamma_{M_i} = (l_{i0} + l_{i1} + ... + l_{ik}) + (m_{i0} + m_{i1} + ... + m_{ij}).$$

Here the degree of the differential polynomial $Q[f, g]$ is given by

$$\gamma_Q = max\{\gamma_{M_i} : 1 \leq i \leq n\}.$$

From (3.17) we can have

$$\begin{aligned}
m(r, Q[f, g]) &= m(r, q_1M_1[f, g] + q_2M_2[f, g] + ... + q_nM_n[f, g]) \\
&\leq m(r, q_1M_1[f, g]) + m(r, q_2M_2[f, g]) + ... + m(r, q_nM_n[f, g]) + O(1) \\
&\leq m(r, q_1) + m(r, M_1[f, g]) + m(r, q_2) + m(r, M_2[f, g]) + ... \\
&\qquad + m(r, q_n) + m(r, M_n[f, g]) + O(1) \\
&\leq \sum_{i=1}^{n} m(r, q_i) + 2l_1m(r, f) + 2m_1m(r, g) + 2l_2m(r, f) + 2m_2m(r, g) \\
&\qquad\qquad + ... + 2l_nm(r, f) + 2m_nm(r, g) + S(r, f) + S(r, g), \\
&\quad \textit{assuming } l_i = (l_{i0} + l_{i1} + ... + l_{ik}) \textit{ and } m_i = (m_{i0} + m_{i1} + ... \\
&\quad + m_{ij}) \textit{ and using } (3.15) \\
&= \sum_{i=1}^{n} m(r, q_i) + 2m(r, f)\sum_{i=1}^{n} l_i + 2m(r, g)\sum_{i=1}^{n} m_i + S(r, f) + S(r, g)
\end{aligned}$$

$$\leq \sum_{i=1}^{n} m(r, q_i) + 2m(r, f)\sum_{i=1}^{n}(l_i + m_i) + 2m(r, g)\sum_{i=1}^{n}(l_i + m_i)$$
$$+ S(r, f) + S(r, g)$$
$$= \sum_{i=1}^{n} m(r, q_i) + 2[m(r, f) + m(r, g)]\sum_{i=1}^{n}(l_i + m_i) + S(r, f) + S(r, g)$$
$$= \sum_{i=1}^{n} m(r, q_i) + 2[m(r, f) + m(r, g)]\sum_{i=1}^{n}\gamma_{M_i} + S(r, f) + S(r, g)$$
$$\leq \sum_{i=1}^{n} m(r, q_i) + 2n\gamma_Q[m(r, f) + m(r, g)] + S(r, f) + S(r, g)$$
$$= 2n\gamma_Q[m(r, f) + m(r, g)] + \sum_{i=1}^{n} m(r, q_i) + S(r, f) + S(r, g).$$

Hence the proof is complete.

**Lemma 3.2.** *Let $Q^*[f, g]$ and $Q[f, g]$ denote differential polynomials in $f$ and $g$ with arbitrary meromorphic coefficients ${q_1}^*, {q_2}^*, ..., {q_n}^*$ and $q_1, q_2, ..., q_k$ respectively. Suppose that*

$$P[f, g] = f^l g^m + a_{l-1}f^{l-1}g^m + b_{m-1}f^l g^{m-1} + a_{l-2}f^{l-2}g^m$$
$$+b_{m-2}f^l g^{m-2} + ... + a_0 g^m + b_0 f^l. \quad (3.18)$$

*If*

$$P[f, g]Q^*[f, g] = Q[f, g], \quad (3.19)$$

*$\gamma_Q \leq (l + m)$, in fact if*

$$Q[f, g] = q_1M_1[f, g] + q_2M_2[f, g] + ... + q_kM_k[f, g] \quad (3.20)$$

*where*

$$M_i[f, g] = f^{\alpha_{i0}}(f^{(1)})^{\alpha_{i1}}...(f^{(j)})^{\alpha_{ij}}g^{\beta_{i0}}(g^{(1)})^{\beta_{i1}}...(g^{(h)})^{\beta_{ih}} \quad for \quad i = 1, 2, ..., k;$$

*with $\alpha_{i0}, \alpha_{i1}, ..., \alpha_{ij}, \beta_{i0}, \beta_{i1}, ..., \beta_{ih}, j, h$ being non-negative integers such that*

$$max\{\alpha_i = \alpha_{i0} + \alpha_{i1} + ... + \alpha_{ij} : 1 \leq i \leq k\} \leq l \quad and$$

$$max\{\beta_i = \beta_{i0} + \beta_{i1} + ... + \beta_{ih} : 1 \leq i \leq k\} \leq m$$

*and on the circle* $|z| = r$, $|f(re^{i\theta})| = |g(re^{i\theta})|$ , *then*

$$m(r, Q^*[f, g]) \leq \sum_{i=1}^{n} m(r, q_i{}^*) + \sum_{i=1}^{k} m(r, q_i) + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g).$$

**Proof.** We first prove the case when

$$a_{l-1} = a_{l-2} = ... = a_0 = 0 = b_{m-1} = b_{m-2} = ... = b_0.$$

In this case we can rewrite (3.19) as

$$f^l g^m Q^*[f, g] = Q[f, g]. \tag{3.21}$$

Now we suppose

$$Q^*[f, g] = q_1^* M_1^*[f, g] + q_2^* M_2^*[f, g] + ... + q_n^* M_n^*[f, g] \tag{3.22}$$

where

$$M_i^*[f, g] = f^{s_{i0}}(f^{(1)})^{s_{i1}}...(f^{(u)})^{s_{iu}} g^{t_{i0}}(g^{(1)})^{t_{i1}}...(g^{(v)})^{t_{iv}} \quad for \quad i = 1, 2, ..., n$$

with $s_{i0}, s_{i1}, ..., s_{iu}, t_{i0}, t_{i1}, ..., t_{iv}, u, v$ being non-negative integers.
First we consider $|f(re^{i\theta})| = |g(re^{i\theta})| > 1$.
Then from (3.20) and (3.21) we have

$$\begin{aligned}
|Q^*[f, g]| &= |f^{-l} g^{-m} Q[f, g]| \\
&= |f^{-l} g^{-m} \sum_{i=1}^{k} q_i M_i[f, g]| \\
&= |\sum_{i=1}^{k} q_i (\frac{f^{(1)}}{f})^{\alpha_{i1}} ... (\frac{f^{(j)}}{f})^{\alpha_{ij}} (\frac{g^{(1)}}{g})^{\beta_{i1}} ... (\frac{g^{(h)}}{f})^{\beta_{ih}} f^{\alpha_i - l} g^{\beta_i - m}| \\
&\leq \sum_{i=1}^{k} |q_i| |\frac{f^{(1)}}{f}|^{\alpha_{i1}} ... |\frac{f^{(j)}}{f}|^{\alpha_{ij}} |\frac{g^{(1)}}{g}|^{\beta_{i1}} ... |\frac{g^{(h)}}{f}|^{\beta_{ih}} |f|^{\alpha_i - l} |g|^{\beta_i - m} \\
&\leq \sum_{i=1}^{k} |q_i| |\frac{f^{(1)}}{f}|^{\alpha_{i1}} ... |\frac{f^{(j)}}{f}|^{\alpha_{ij}} |\frac{g^{(1)}}{g}|^{\beta_{i1}} ... |\frac{g^{(h)}}{f}|^{\beta_{ih}}, \\
&\qquad since \quad |f(re^{i\theta})| = |g(re^{i\theta})| > 1 \quad and \quad \alpha_i - l \leq 0, \quad \beta_i - m \leq 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{1}{2\pi}\int_{|f|=|g|>1}\log^{+}|Q^{*}[f,g]|d\theta \\
&\leq \frac{1}{2\pi}\int_{0}^{2\pi}\log^{+}(\sum_{i=1}^{k}|q_i||\frac{f^{(1)}}{f}|^{\alpha_{i1}}...|\frac{f^{(j)}}{f}|^{\alpha_{ij}}|\frac{g^{(1)}}{g}|^{\beta_{i1}}...|\frac{g^{(h)}}{f}|^{\beta_{ih}})d\theta \\
&\leq \sum_{i=1}^{k}\frac{1}{2\pi}\int_{0}^{2\pi}\log^{+}(|q_i||\frac{f^{(1)}}{f}|^{\alpha_{i1}}...|\frac{f^{(j)}}{f}|^{\alpha_{ij}}|\frac{g^{(1)}}{g}|^{\beta_{i1}}...|\frac{g^{(h)}}{f}|^{\beta_{ih}})d\theta + O(1) \\
&= \sum_{i=1}^{k}\frac{1}{2\pi}\int_{0}^{2\pi}\log^{+}|q_i|d\theta + \sum_{i=1}^{k}(\frac{1}{2\pi}\int_{0}^{2\pi}\log^{+}|\frac{f^{(1)}}{f}|^{\alpha_{i1}}d\theta + ... \\
&\qquad ... + \frac{1}{2\pi}\int_{0}^{2\pi}\log^{+}|\frac{f^{(j)}}{f}|^{\alpha_{ij}}d\theta + \frac{1}{2\pi}\int_{0}^{2\pi}\log^{+}|\frac{g^{(1)}}{g}|^{\beta_{i1}}d\theta + ... \\
&\qquad ... + \frac{1}{2\pi}\int_{0}^{2\pi}\log^{+}|\frac{g^{(h)}}{f}|^{\beta_{ih}}d\theta) + O(1) \\
&= \sum_{i=1}^{k}m(r,q_i) + \sum_{i=1}^{k}[\alpha_{i1}m(r,\frac{f^{(1)}}{f}) + ... + \alpha_{ij}m(r,\frac{f^{(j)}}{f}) \\
&\qquad + \beta_{i1}m(r,\frac{g^{(1)}}{g}) + ... + \beta_{ih}m(r,\frac{g^{(h)}}{g})] + O(1) \\
&= \sum_{i=1}^{k}m(r,q_i) + \sum_{i=1}^{k}[\alpha_{i1}S(r,f) + ... + \alpha_{ij}S(r,f) \\
&\qquad + \beta_{i1}S(r,g) + ... + \beta_{ih}S(r,g)] + O(1), \\
&\qquad\qquad using\ Milloux's\ Theorem \\
&= \sum_{i=1}^{k}m(r,q_i) + S(r,f) + S(r,g). \qquad (3.23)
\end{aligned}$$

Next we consider $|f(re^{i\theta})| = |g(re^{i\theta})| \leq 1$. From (3.22) we have

$$\begin{aligned}
|Q^{*}[f,g]| &= |\sum_{i=1}^{n}q_i^{*}M_i^{*}[f,g]| \\
&= |\sum_{i=1}^{n}q_i^{*}f^{s_{i0}}(f^{(1)})^{s_{i1}}...(f^{(u)})^{s_{iu}}g^{t_{i0}}(g^{(1)})^{t_{i1}}...(g^{(v)})^{t_{iv}}|
\end{aligned}$$

$$\leq \sum_{i=1}^{n} |q_i^*|| \frac{f^{(1)}}{f}|^{s_{i1}} ... |\frac{f^{(u)}}{f}|^{s_{iu}} |\frac{g^{(1)}}{g}|^{t_{i1}} ... |\frac{g^{(v)}}{g}|^{t_{iv}}.$$

Hence,

$$\begin{aligned}
&\frac{1}{2\pi}\int_{|f|=|g|\leq 1} \log^+ |Q^*[f,g]| d\theta \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} \log^+ (\sum_{i=1}^{n} |q_i^*|| \frac{f^{(1)}}{f}|^{s_{i1}} ... |\frac{f^{(u)}}{f}|^{s_{iu}} |\frac{g^{(1)}}{g}|^{t_{i1}} ... |\frac{g^{(v)}}{g}|^{t_{iv}}) d\theta \\
&\leq \sum_{i=1}^{n} \frac{1}{2\pi}\int_0^{2\pi} \log^+ (|q_i^*|| \frac{f^{(1)}}{f}|^{s_{i1}} ... |\frac{f^{(u)}}{f}|^{s_{iu}} |\frac{g^{(1)}}{g}|^{t_{i1}} ... |\frac{g^{(v)}}{g}|^{t_{iv}}) d\theta + O(1) \\
&= \sum_{i=1}^{n} \frac{1}{2\pi}\int_0^{2\pi} \log^+ |q_i^*| d\theta + \sum_{i=1}^{n} (\frac{1}{2\pi}\int_0^{2\pi} \log^+ |\frac{f^{(1)}}{f}|^{s_{i1}} d\theta + ... \\
&\qquad ... + \frac{1}{2\pi}\int_0^{2\pi} \log^+ |\frac{f^{(u)}}{f}|^{s_{iu}} d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log^+ |\frac{g^{(1)}}{g}|^{t_{i1}} d\theta + ... \\
&\qquad ... + \frac{1}{2\pi}\int_0^{2\pi} \log^+ |\frac{g^{(v)}}{g}|^{t_{iv}} d\theta) + O(1) \\
&= \sum_{i=1}^{n} m(r, q_i^*) + \sum_{i=1}^{n} (s_{i1} m(r, \frac{f^{(1)}}{f}) + ... + s_{iu} m(r, \frac{f^{(u)}}{f}) \\
&\qquad + t_{i1} m(r, \frac{g^{(1)}}{g}) + ... + t_{iv} m(r, \frac{g^{(v)}}{g})) + O(1) \\
&= \sum_{i=1}^{n} m(r, q_i^*) + \sum_{i=1}^{n} (s_{i1} S(r,f) + ... + s_{iu} S(r,f) \\
&\qquad + t_{i1} S(r,g) + ... + t_{iv} S(r,g)) + O(1), \\
&\qquad\qquad\qquad\qquad \textit{using Milloux's Theorem} \\
&= \sum_{i=1}^{n} m(r, q_i^*) + S(r,f) + S(r,g). \qquad (3.24)
\end{aligned}$$

Adding (3.23) and (3.24) we get

$$\begin{aligned}
&\frac{1}{2\pi}\int_{|f|=|g|>1} \log^+ |Q^*[f,g]| d\theta + \frac{1}{2\pi}\int_{|f|=|g|\leq 1} \log^+ |Q^*[f,g]| d\theta \\
&\leq \sum_{i=1}^{n} m(r, q_i^*) + \sum_{i=1}^{k} m(r, q_i) + S(r,f) + S(r,g)
\end{aligned}$$

i.e.,

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+|Q^*[f,g]|d\theta \leq \sum_{i=1}^{n} m(r,q_i^*) + \sum_{i=1}^{k} m(r,q_i) + S(r,f) + S(r,g)$$

i.e.,

$$m(r,Q^*[f,g]) \leq \sum_{i=1}^{n} m(r,q_i^*) + \sum_{i=1}^{k} m(r,q_i) + S(r,f) + S(r,g).$$

Now we consider the general case when $a_{l-1}, a_{l-2}, ..., a_0; b_{m-1}, b_{m-2}, ..., b_0$ are any arbitrary meromorphic functions with smaller growth than $f$ and $g$ respectively. Then from (3.19) we can write

$$f^l g^m[(f^l g^m + a_{l-1}f^{l-1}g^m + b_{m-1}f^l g^{m-1} + ... + a_0 g^m + b_0 f^l)Q^*[f,g]] = f^l g^m Q[f,g]$$

or, $f^l g^m R^*[f,g] = R[f,g]$, say where $R^*[f,g]$ and $R[f,g]$ are differential polynomials in $f$ and $g$ with meromorphic coefficients $q_1^*(1 + a_{l-1} + b_{m-1} + ... + a_0 + b_0)$, $q_2^*(1 + a_{l-1} + b_{m-1} + ... + a_0 + b_0)$, ... , $q_n^*(1 + a_{l-1} + b_{m-1} + ... + a_0 + b_0)$ and $q_1$, $q_2$, ... , $q_k$ respectively.
Hence by first case

$$\begin{aligned} m(r,R^*[f,g]) &\leq \sum_{i=1}^{n} m(r,q_i^*(1 + a_{l-1} + b_{m-1} + ... + a_0 + b_0)) \\ &\quad + \sum_{i=1}^{k} m(r,q_i) + S(r,f) + S(r,g) \\ &\leq \sum_{i=1}^{n} m(r,q_i^*) + \sum_{i=1}^{k} m(r,q_i) + S(r,f) + S(r,g) \end{aligned}$$

So,

$$\begin{aligned} m(r,Q^*[f,g]) &\leq m(r,R^*[f,g]) + m(r,\frac{1}{P[f,g]}) \\ &\leq \sum_{i=1}^{n} m(r,q_i^*) + \sum_{i=1}^{k} m(r,q_i) + m(r,\frac{1}{P[f,g]}) + S(r,f) + S(r,g) \end{aligned}$$

Hence the proof is complete.

**Lemma 3.3.** *Suppose that $M[f,g]$ is a monomial in $f$ and $g$. If $f$ and $g$ has a* *pole at $z = z_0$ of order $p$ and $q$ respectively, then $z_0$ is a pole of $M[f,g]$ of order at most $(p+q-2)\gamma_M + \Gamma_M$.*

**Proof.** Let

$$M[f,g] = (f(z))^{l_0}(f^{(1)}(z))^{l_1}...(f^{(k)}(z))^{l_k}(g(z))^{m_0}(g^{(1)}(z))^{m_1}...(g^{(j)}(z))^{m_j}$$

where $l_0$, $l_1$, ... , $l_k$, $m_0$, $m_1$, ... , $m_j$ are non-negative integers;

$$\gamma_M = (l_0 + l_1 + ... + l_k) + (m_0 + m_1 + ... + m_j)$$

is the degree of the monomial and

$$\Gamma_M = \{l_0 + 2l_1 + 3l_2 + ... + (k+1)l_k\} + \{m_0 + 2m_1 + 3m_2 + ... + (j+1)m_j\}$$

is the weight of the monomial.

Now since $z_0$ is a pole of $f$ and $g$ of order $p$ and $q$ respectively, $z_0$ will be a pole of $M[f,g]$ of order

$$\begin{aligned}
&= \{pl_0 + (p+1)l_1 + ... + (p+k)l_k\} + \{qm_0 + (q+1)m_1 + ... + (q+j)m_j\} \\
&= \{p(l_0 + l_1 + ... + l_k) + (l_1 + 2l_2 + ... + kl_k)\} + \{q(m_0 + m_1 + ... + m_j) \\
&\qquad + (m_1 + 2m_2 + ... + jm_j)\} \\
&= [(p-1)(l_0 + l_1 + ... + l_k) + \{l_0 + 2l_1 + 3l_2 + ... + (k+1)l_k\}] \\
&\qquad + [(q-1)(m_0 + m_1 + ... + m_j) + \{m_0 + 2m_1 + 3m_2 + ... + (j+1)m_j\}] \\
&\leq (p-1)\gamma_M + (q-1)\gamma_M + \{l_0 + 2l_1 + 3l_2 + ... + (k+1)l_k\} + \{m_0 + 2m_1 \\
&\qquad + 3m_2 + ... + (j+1)m_j\} \\
&= (p+q-2)\gamma_M + \Gamma_M.
\end{aligned}$$

Hence the proof is complete.

**Lemma 3.4.** *Suppose that $Q[f,g]$ is a differential polynomial in $f$ and $g$. Let $z_0$ be a pole of $f$ and $g$ of order $p$ and $q$ respectively and not a zero or a pole of coefficients of $Q[f,g]$. Then $z_0$ is a pole of $Q[f,g]$ of order at most $(p+q-1)\gamma_Q + (\Gamma_Q - \gamma_Q)$.*

**Proof.** Let

$$Q[f,g] = q_1M_1[f,g] + q_2M_2[f,g] + ... + q_nM_n[f,g]$$

where each

$$M_i[f,g] = (f(z))^{l_{i0}}(f^{(1)}(z))^{l_{i1}}...(f^{(k)}(z))^{l_{ik}}(g(z))^{m_{i0}}(g^{(1)}(z))^{m_{i1}}...(g^{(j)}(z))^{m_{ij}}$$

is a monomial in $f$ and $g$ of degree

$$\gamma_{M_i} = (l_{i0} + l_{i1} + ... + l_{ik}) + (m_{i0} + m_{i1} + ... + m_{ij})$$

and of weight

$$\Gamma_{M_i} = \{l_{i0} + 2l_{i1} + ... + (k+1)l_{ik}\} + \{m_{i0} + 2m_{i1} + ... + (j+1)m_{ij}\}.$$

Here the degree and weight of the differential polynomial $Q[f,g]$ is given by

$$\gamma_Q = max\{\gamma_{M_i} : 1 \leq i \leq n\}$$

and

$$\Gamma_Q = max\{\Gamma_{M_i} : 1 \leq i \leq n\}$$

respectively.
Now since $z_0$ is a pole of $f$ and $g$ of order $p$ and $q$ respectively and not a zero or a pole of the coefficients of $Q[f,g]$, by using the Lemma 3.3 we can say that $z_0$ is a pole of $Q[f,g]$ of order

$$\begin{aligned}
&\leq \max_{1\leq i\leq n}\{(p+q-2)\gamma_{M_i} + \Gamma_{M_i}\} \\
&\leq \max_{1\leq i\leq n}\{(p+q-2)\gamma_{M_i}\} + \max_{1\leq i\leq n}\{\Gamma_{M_i}\} \\
&= (p+q-2)\gamma_Q + \Gamma_Q \\
&= (p+q-1)\gamma_Q + (\Gamma_Q - \gamma_Q).
\end{aligned}$$

Hence the proof is complete.

**Theorem 3.2.** *Let $f$ and $g$ be transcendental meromorphic functions in the plane and $Q_1[f,g]$, $Q_2[f,g]$ be differential polynomials in $f$ and $g$ satisfying $Q_1[f,g] \not\equiv 0$, $Q_2[f,g] \not\equiv 0$ with $Q_1[f,g]$ and $Q_2[f,g]$ consisting of $n$ arbitrary meromorphic coefficients which are small functions of $f$ and $g$ and $P[f,g]$ be given by* (3.18). *If*

$$F = P[f,g]Q_1[f,g] + Q_2[f,g] \tag{3.25}$$

*and on the circle $|z| = r$, $|f(re^{i\theta})| = |g(re^{i\theta})|$, then*

$$\begin{aligned}
(l - 2n\gamma_{Q_2})T(r,f) + (m - 2n\gamma_{Q_2})T(r,g) \leq{}& \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{P[f,g]}) + (\Gamma_{Q_2} \\
&- 2\gamma_{Q_2})[\bar{N}(r,f) + \bar{N}(r,g)] + S(r,f) \\
&+ S(r,g).
\end{aligned}$$

**Proof.** Without loss of generality we may suppose $l > 2n\gamma_{Q_2}$ and $m > 2n\gamma_{Q_2}$.
From (3.25) we have

$$1 = \frac{1}{F}P[f,g]Q_1[f,g] + \frac{1}{F}Q_2[f,g]$$

i.e.,

$$F' = \frac{F'}{F} P[f,g]Q_1[f,g] + \frac{F'}{F} Q_2[f,g]. \tag{3.26}$$

Again differentiating (3.25) we have

$$F' = (P[f,g])' Q_1[f,g] + P[f,g](Q_1[f,g])' + (Q_2[f,g])'. \tag{3.27}$$

Now comparing (3.26) and (3.27) we have

$$\frac{F'}{F} P[f,g]Q_1[f,g] + \frac{F'}{F} Q_2[f,g] = (P[f,g])' Q_1[f,g] + P[f,g](Q_1[f,g])' + (Q_2[f,g])'$$

$$i.e., \ \frac{F'}{F} P[f,g]Q_1[f,g] - (P[f,g])' Q_1[f,g] - P[f,g](Q_1[f,g])' = (Q_2[f,g])' - \frac{F'}{F} Q_2[f,g]$$

$$i.e., \quad P[f,g]\{\frac{F'}{F} Q_1[f,g] - \frac{(P[f,g])'}{P[f,g]} Q_1[f,g] - (Q_1[f,g])'\} = (Q_2[f,g])' - \frac{F'}{F} Q_2[f,g]$$

i.e.,

$$P[f,g]Q^*[f,g] = Q[f,g] \tag{3.28}$$

where

$$Q^*[f,g] = \frac{F'}{F} Q_1[f,g] - \frac{(P[f,g])'}{P[f,g]} Q_1[f,g] - (Q_1[f,g])' \tag{3.29}$$

and

$$Q[f,g] = (Q_2[f,g])' - \frac{F'}{F} Q_2[f,g]. \tag{3.30}$$

If $Q^*[f,g] = 0$, then from (3.28) we have

$$Q[f,g] = 0$$

$$i.e., \quad (Q_2[f,g])' - \frac{F'}{F} Q_2[f,g] = 0, \qquad using \ (3.30)$$

$$i.e., \quad \frac{(Q_2[f,g])'}{Q_2[f,g]} = \frac{F'}{F}.$$

Integrating we get

$$F = C_1 Q_2[f,g]$$

$$i.e., \quad P[f,g]Q_1[f,g] + Q_2[f,g] = C_1 Q_2[f,g], \qquad using \ (3.25)$$

$$i.e., \quad P[f,g]Q_1[f,g] = CQ_2[f,g], \tag{3.31}$$

assuming $C_1 - 1 = C(\neq 0)$ is a finite complex number.

Now using the Lemma 3.2 and assuming that the coefficients of $Q_1[f,g]$ and $Q_2[f,g]$ are small functions of $f$ and $g$, we get

$$m(r, Q_1[f,g]) = S(r,f) + S(r,g). \tag{3.32}$$

Again from (3.31) we have

$$\begin{aligned} m(r, P[f,g]) &= m(r, C\frac{Q_2[f,g]}{Q_1[f,g]}) \\ &\leq m(r, Q_2[f,g]) + m(r, \frac{1}{Q_1[f,g]}) + O(1). \end{aligned} \tag{3.33}$$

Also from (3.15) we can write

$$m(r, P[f,g]) = 2l \times m(r,f) + 2m \times m(r,g) + S(r,f) + S(r,g). \tag{3.34}$$

Again if $Q_2[f,g]$ consists of $n$ arbitrary meromorphic coefficients then applying Lemma 3.1 and using the hypothesis that the coefficients of $Q_2[f,g]$ to be small functions of $f$ and $g$, we get

$$m(r, Q_2[f,g]) \leq 2n\gamma_{Q_2}[m(r,f) + m(r,g)] + S(r,f) + S(r,g). \tag{3.35}$$

Also, by the First Fundamental Theorem we have

$$T(r, \frac{1}{Q_1[f,g]}) = T(r, Q_1[f,g]) + O(1)$$

$$i.e., \quad m(r, \frac{1}{Q_1[f,g]}) + N(r, \frac{1}{Q_1[f,g]}) = m(r, Q_1[f,g]) + N(r, Q_1[f,g]) + O(1)$$

$$i.e., \ m(r, \frac{1}{Q_1[f,g]}) + N(r, \frac{1}{Q_1[f,g]}) = N(r, Q_1[f,g]) + S(r,f) + S(r,g), \quad using\ (3.32)$$

$$i.e., \quad m(r, \frac{1}{Q_1[f,g]}) = N(r, Q_1[f,g]) - N(r, \frac{1}{Q_1[f,g]}) + S(r,f) + S(r,g). \tag{3.36}$$

It is clear that a pole of $Q_1[f,g]$ is either a pole of $f$ or $g$, or a pole of the coefficients of $Q_1[f,g]$.

Now suppose that $z_0$ is a pole of $f$ and $g$ of order $p$ and $q$ respectively and $z_0$ is not a zero or a pole of the coefficients of $P[f,g]$, $Q_1[f,g]$ and $Q_2[f,g]$. Then from (3.18) we can say that $z_0$ is a pole of $P[f,g]$ of order $pl + qm$.

Also from Lemma 3.4 we can say that $z_0$ is a pole of $Q_2[f,g]$ of order at most

$(p+q-1)\gamma_{Q_2} + (\Gamma_{Q_2} - \gamma_{Q_2})$.
Now from (3.31) we have

$$Q_1[f,g] = C\frac{Q_2[f,g]}{P[f,g]}$$

and hence, we can say that $z_0$ is a pole of $Q_1[f,g]$ of order at most

$$\begin{aligned}&(p+q-1)\gamma_{Q_2} + (\Gamma_{Q_2} - \gamma_{Q_2}) - (pl+qm)\\&= (\Gamma_{Q_2} - 2\gamma_{Q_2}) - p(l-\gamma_{Q_2}) - q(m-\gamma_{Q_2}).\end{aligned}$$

Thus we obtain

$$\begin{aligned}&N(r, Q_1[f,g]) - N(r, \frac{1}{Q_1[f,g]})\\&\leq (\Gamma_{Q_2} - 2\gamma_{Q_2})[\bar{N}(r,f) + \bar{N}(r,g)] - (l-\gamma_{Q_2})N(r,f)\\&\qquad - (m-\gamma_{Q_2})N(r,g) + S(r,f) + S(r,g).\end{aligned} \tag{3.37}$$

Now using (3.37) in (3.36) we get

$$\begin{aligned}m(r, \frac{1}{Q_1[f,g]}) &\leq (\Gamma_{Q_2} - 2\gamma_{Q_2})[\bar{N}(r,f) + \bar{N}(r,g)] - (l-\gamma_{Q_2})N(r,f)\\&\qquad - (m-\gamma_{Q_2})N(r,g) + S(r,f) + S(r,g).\end{aligned} \tag{3.38}$$

Now from (3.33), (3.34), (3.35) and (3.38) we obtain

$$\begin{aligned}2l \times m(r,f) + 2m \times m(r,g) &\leq 2n\gamma_{Q_2}[m(r,f) + m(r,g)] + (\Gamma_{Q_2} - 2\gamma_{Q_2})[\bar{N}(r,f)\\&\quad + \bar{N}(r,g)] - (l-\gamma_{Q_2})N(r,f)\\&\quad - (m-\gamma_{Q_2})N(r,g) + S(r,f) + S(r,g)\\ i.e.,\ l \times m(r,f) + m \times m(r,g) &\leq 2n\gamma_{Q_2}[m(r,f) + m(r,g)] + (\Gamma_{Q_2} - 2\gamma_{Q_2})[\bar{N}(r,f)\\&\quad + \bar{N}(r,g)] - (l-2n\gamma_{Q_2})N(r,f)\\&\quad - (m-2n\gamma_{Q_2})N(r,g) + S(r,f) + S(r,g)\\ i.e.,\ l[m(r,f) + N(r,f)] + m[m(r,g) + N(r,g)] &\leq 2n\gamma_{Q_2}[m(r,f) + N(r,f)]\\&\quad + 2n\gamma_{Q_2}[m(r,g) + N(r,g)]\\&\quad + (\Gamma_{Q_2} - 2\gamma_{Q_2})[\bar{N}(r,f)\\&\quad + \bar{N}(r,g)] + S(r,f) + S(r,g)\\ i.e.,\ lT(r,f) + mT(r,g) &\leq 2n\gamma_{Q_2}T(r,f) + 2n\gamma_{Q_2}T(r,g) + (\Gamma_{Q_2} - 2\gamma_{Q_2})[\bar{N}(r,f)\\&\quad + \bar{N}(r,g)] + S(r,f) + S(r,g)\\ i.e.,\ (l-2n\gamma_{Q_2})T(r,f) + (m-2n\gamma_{Q_2})T(r,g) &\leq (\Gamma_{Q_2} - 2\gamma_{Q_2})[\bar{N}(r,f)\\&\quad + \bar{N}(r,g)] + S(r,f) + S(r,g)\end{aligned}$$

$$i.e., \ (l-2n\gamma_{Q_2})T(r,f)+(m-2n\gamma_{Q_2})T(r,g)\leq \bar{N}(r,\frac{1}{F})+\bar{N}(r,\frac{1}{P[f,g]}) + (\Gamma_{Q_2}-2\gamma_{Q_2})[\bar{N}(r,f) \quad +\bar{N}(r,g)]+S(r,f)+S(r,g),$$

which is the required result.
Next we suppose that $Q^*[f,g]\not\equiv 0$. Then from (3.28) we have $Q[f,g]\not\equiv 0$.
Now a monomial in $f$ and $g$ is an expression of the form

$$M[f,g]=(f(z))^{l_0}(f^{(1)}(z))^{l_1}...(f^{(k)}(z))^{l_k}(g(z))^{m_0}(g^{(1)}(z))^{m_1}...(g^{(j)}(z))^{m_j}$$

with degree

$$\gamma_M=(l_0+l_1+...+l_k)+(m_0+m_1+...+m_j),$$

where $l_0$, $l_1$, ... , $l_k$, $m_0$, $m_1$, ... , $m_j$ are non-negative integers.
On differentiation, we get

$$\begin{aligned}
&(M[f,g])'\\
&=l_0(f(z))^{l_0-1}(f^{(1)}(z))^{l_1+1}...(f^{(k)}(z))^{l_k}(g(z))^{m_0}(g^{(1)}(z))^{m_1}...(g^{(j)}(z))^{m_j}\\
&+l_1(f(z))^{l_0}(f^{(1)}(z))^{l_1-1}...(f^{(k)}(z))^{l_k}(g(z))^{m_0}(g^{(1)}(z))^{m_1}...(g^{(j)}(z))^{m_j}+...\\
&...+l_k(f(z))^{l_0}(f^{(1)}(z))^{l_1}...(f^{(k)}(z))^{l_k-1}f^{(k+1)}(z)(g(z))^{m_0}(g^{(1)}(z))^{m_1}...(g^{(j)}(z))^{m_j}\\
&+m_0(f(z))^{l_0}(f^{(1)}(z))^{l_1}...(f^{(k)}(z))^{l_k}(g(z))^{m_0-1}(g^{(1)}(z))^{m_1+1}...(g^{(j)}(z))^{m_j}\\
&+m_1(f(z))^{l_0}(f^{(1)}(z))^{l_1}...(f^{(k)}(z))^{l_k}(g(z))^{m_0}(g^{(1)}(z))^{m_1-1}...(g^{(j)}(z))^{m_j}+...\\
&...+m_j(f(z))^{l_0}(f^{(1)}(z))^{l_1}...(f^{(k)}(z))^{l_k}(g(z))^{m_0}(g^{(1)}(z))^{m_1}...(g^{(j)}(z))^{m_j-1}g^{(j+1)}(z).
\end{aligned}$$

Hence

$$\begin{aligned}
\gamma_{M'}&=max\{[(l_0-1)+(l_1+1)+...+l_k+m_0+m_1+...+m_j],[l_0+(l_1-1)\\
&+(l_2+1)+...+l_k+m_0+m_1+...+m_j],...[l_0+l_1+...+(l_k-1)+1\\
&+m_0+m_1+...+m_j],[l_0+l_1+...+l_k+(m_0-1)+(m_1+1)+...+m_j],\\
&[l_0+l_1+...+l_k+m_0+(m_1-1)+(m_2+1)...+m_j],[l_0+l_1+...+l_k\\
&+m_0+m_1+...+(m_j-1)+1]\}\\
&=\max\{\gamma_M,\gamma_M,...,\gamma_M\}\\
&=\gamma_M.
\end{aligned}$$

Thus we can write $\gamma_{Q_2}=\gamma_{(Q_2)'}$. Hence from (3.30) we have $\gamma_Q=\gamma_{Q_2}$.
Now using Lemma 3.2 on (3.28) and assuming that the coefficients of $Q^*[f,g]$ and $Q[f,g]$ are small functions of $f$ and $g$, we get

$$m(r,Q^*[f,g])=S(r,f)+S(r,g). \tag{3.39}$$

Again from (3.28) we have

$$m(r, P[f,g]) = m(r, \frac{Q[f,g]}{Q^*[f,g]})$$
$$\leq m(r, Q[f,g]) + m(r, \frac{1}{Q^*[f,g]}). \quad (3.40)$$

Using (3.30), Lemma 3.1, Milloux's theorem and the fact that the coefficients of $Q_2[f,g]$) are small functions of $f$ and $g$ we have

$$\begin{aligned} m(r, Q[f,g]) &= m(r, (Q_2[f,g])' - \frac{F'}{F} Q_2[f,g]) \\ &= m(r, Q_2[f,g]\{\frac{(Q_2[f,g])'}{Q_2[f,g]} - \frac{F'}{F}\}) \\ &\leq m(r, Q_2[f,g]) + m(r, \frac{(Q_2[f,g])'}{Q_2[f,g]}) + m(r, -\frac{F'}{F}) + O(1) \\ &= m(r, Q_2[f,g]) + S(r,f) + S(r,g) \\ &\leq 2n\gamma_{Q_2}[m(r,f) + m(r,g)] + S(r,f) + S(r,g). \end{aligned} \quad (3.41)$$

Also, by the First Fundamental Theorem we have

$$T(r, \frac{1}{Q^*[f,g]}) = T(r, Q^*[f,g]) + O(1)$$

$$\textit{i.e.,} \quad m(r, \frac{1}{Q^*[f,g]}) + N(r, \frac{1}{Q^*[f,g]}) = m(r, Q^*[f,g]) + N(r, Q^*[f,g]) + O(1)$$

$$\textit{i.e.,} \quad m(r, \frac{1}{Q^*[f,g]}) + N(r, \frac{1}{Q^*[f,g]}) = N(r, Q^*[f,g]) + S(r,f) + S(r,g), \quad \textit{using } (3.39)$$

$$\textit{i.e.,} \quad m(r, \frac{1}{Q^*[f,g]}) = N(r, Q^*[f,g]) - N(r, \frac{1}{Q^*[f,g]}) + S(r,f) + S(r,g). \quad (3.42)$$

Obviously, from (3.29) we can say that the poles of $Q^*[f,g]$ occurs possibly only from the zeros of $F$ and $P[f,g]$, the poles of $f$ and $g$ and the zeros and poles of the coefficients.

Now suppose that $z_0$ is a pole of $f$ and $g$ of order $p$ and $q$ respectively and not a zero or a pole of the coefficients of $P[f,g]$, $Q_1[f,g]$ and $Q_2[f,g]$.

Then from (3.18) we can say that $z_0$ is a pole of $P[f,g]$ of order $pl + qm$.

Also by using Lemma 3.4 on (3.30) we can say that $z_0$ is a pole of $Q[f,g]$ of order

at most $(p+q-1)\gamma_{Q_2}+(\Gamma_{Q_2}-\gamma_{Q_2})$.
Now from (3.28) we have

$$Q^*[f,g]=\frac{Q[f,g]}{P[f,g]}$$

and hence we can say that if $z_0$ is a pole of $Q^*[f,g]$, it will be a pole of $Q^*[f,g]$ of order at most

$$\begin{aligned}&(p+q-1)\gamma_{Q_2}+(\Gamma_{Q_2}-\gamma_{Q_2})-(pl+qm)\\&=(\Gamma_{Q_2}-2\gamma_{Q_2})-p(l-\gamma_{Q_2})-q(m-\gamma_{Q_2}).\end{aligned}$$

And if $z_0$ is not a pole of $Q^*[f,g]$, we have from (3.28)

$$\frac{1}{Q^*[f,g]}=\frac{P[f,g]}{Q[f,g]}$$

and hence $z_0$ will be a zero of $Q^*[f,g]$ of order at least

$$\begin{aligned}&(pl+qm)-(p+q-1)\gamma_{Q_2}+(\Gamma_{Q_2}-\gamma_{Q_2})\\&=p(l-\gamma_{Q_2})+q(m-\gamma_{Q_2})-(\Gamma_{Q_2}-2\gamma_{Q_2}).\end{aligned}$$

Thus we obtain

$$\begin{aligned}N(r,Q^*[f,g])-N(r,\frac{1}{Q^*[f,g]})\leq \bar{N}(r,\frac{1}{F})+\bar{N}(r,\frac{1}{P[f,g]})+(\Gamma_{Q_2}\\-2\gamma_{Q_2})[\bar{N}(r,f)+\bar{N}(r,g)]-(l-\gamma_{Q_2})N(r,f)\\-(m-\gamma_{Q_2})N(r,g)+S(r,f)+S(r,g).\quad(3.43)\end{aligned}$$

Now using (3.43) in (3.42) we get

$$\begin{aligned}m(r,\frac{1}{Q^*[f,g]})\leq \bar{N}(r,\frac{1}{F})+\bar{N}(r,\frac{1}{P[f,g]})+(\Gamma_{Q_2}-2\gamma_{Q_2})[\bar{N}(r,f)+\bar{N}(r,g)]\\-(l-\gamma_{Q_2})N(r,f)-(m-\gamma_{Q_2})N(r,g)+S(r,f)+S(r,g).\quad(3.44)\end{aligned}$$

Now from (3.34), (3.40), (3.41) and (3.44) we obtain

$$\begin{aligned}2l\times m(r,f)+2m\times m(r,g)\leq 2n\gamma_{Q_2}[m(r,f)+m(r,g)]+\bar{N}(r,\frac{1}{F})+\bar{N}(r,\frac{1}{P[f,g]})\\+(\Gamma_{Q_2}-2\gamma_{Q_2})[\bar{N}(r,f)+\bar{N}(r,g)]-(l-\gamma_{Q_2})N(r,f)\\-(m-\gamma_{Q_2})N(r,g)+S(r,f)+S(r,g)\end{aligned}$$

$$i.e.,\ l \times m(r,f) + m \times m(r,g) \leq 2n\gamma_{Q_2}[m(r,f)+m(r,g)] + \bar{N}(r,\frac{1}{F}) + \bar{N}(r,\frac{1}{P[f,g]})$$
$$+(\Gamma_{Q_2} - 2\gamma_{Q_2})[\bar{N}(r,f)+\bar{N}(r,g)] - (l-2n\gamma_{Q_2})N(r,f)$$
$$-(m-2n\gamma_{Q_2})N(r,g) + S(r,f) + S(r,g)$$
$$i.e.,\ l[m(r,f)+N(r,f)] + m[m(r,g)+N(r,g)] \leq \bar{N}(r,\frac{1}{F}) + \bar{N}(r,\frac{1}{P[f,g]})$$
$$+2n\gamma_{Q_2}[m(r,f)+N(r,f)] + 2n\gamma_{Q_2}[m(r,g)+N(r,g)]$$
$$+(\Gamma_{Q_2} - 2\gamma_{Q_2})[\bar{N}(r,f)+\bar{N}(r,g)] + S(r,f) + S(r,g)$$
$$i.e.,\ lT(r,f) + mT(r,g) \leq \bar{N}(r,\frac{1}{F}) + \bar{N}(r,\frac{1}{P[f,g]}) + 2n\gamma_{Q_2}T(r,f) + 2n\gamma_{Q_2}T(r,g)$$
$$+(\Gamma_{Q_2} - 2\gamma_{Q_2})[\bar{N}(r,f)+\bar{N}(r,g)] + S(r,f) + S(r,g)$$
$$i.e.,\ (l-2n\gamma_{Q_2})T(r,f) + (m-2n\gamma_{Q_2})T(r,g) \leq \bar{N}(r,\frac{1}{F}) + \bar{N}(r,\frac{1}{P[f,g]})$$
$$+ (\Gamma_{Q_2} - 2\gamma_{Q_2})[\bar{N}(r,f)+\bar{N}(r,g)] + S(r,f) + S(r,g).$$

Hence the proof is complete.

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