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FUZZY β-CONNECTEDNESS IN THE FUZZY TOPOLOGICAL SPACES

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Abstract: This paper investigates the concept of fuzzy β-connectedness in fuzzy topological spaces by introducing fuzzy β-separated sets and related constructs. We define fuzzy β-disconnected spaces and fuzzy β-connected spaces using fuzzy β-open sets. Several theorems are established to characterize the behavior and properties of these notions. We also examine the preservation of fuzzy β-connectedness under M-fuzzy β-continuous and fuzzy β-open mappings. Illustrative examples are provided to demonstrate the theoretical developments and their implications in the broader context of fuzzy topological spaces.

Keywords and Phrases: Fuzzy topological spaces, fuzzy β-open sets, fuzzy β-separated sets, fuzzy β-disconnected sets, fuzzy β-disconnected spaces.

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1. Introduction

There are approaches such as fuzzy sets [13], intuitionistic fuzzy sets [3], vague sets [8], and rough sets [12] which can be treated as mathematical tools to avert obstacles dealing with ambiguous data. The theory of fuzzy topological spaces was introduced and developed by Chang [6] and since then various notions in classical topology have been extended to fuzzy topological spaces. The concept of fuzzy β-open sets was introduced by Monseb [1] and studied also by Allam and Hakkim [2]. Among these developments, the concept of fuzzy connectedness has garnered attention as a natural extension of classical connectedness in topological spaces. Chaudhuri and Das [7] introduced and investigated fuzzy connected sets, establishing foundational properties and characterizations of connectedness in fuzzy topology. Their work laid the groundwork for deeper investigations into the behavior of connected components and continuity under fuzziness. Building upon these ideas, the concept of fuzzy β-connectedness has emerged as a refinement of fuzzy connectedness.

Katsaras [9] investigated fuzzy proximity spaces, providing a formal framework to study "closeness" under fuzziness. Mashhour, Abd El-Monsef, and El-Deeb's [10] early work on pre-open sets and pre-continuous mappings laid foundational ground for generalized openness and continuity concepts, which later influenced studies on β-open sets and β-continuous mappings that relax strict openness and continuity conditions in both classical and fuzzy settings. Mukherjee and Samanta [11] specifically extended β-open set theory into fuzzy topological spaces by characterizing fuzzy β-open sets and defining fuzzy β-continuity via fuzzy closure and interior operators, thereby enriching the structural understanding of generalized fuzzy mappings and their relationships with other fuzzy continuity concepts.

Recent contributions have also introduced generalized notions in fuzzy set theory, including novel types of coincidence and quasi-coincidence relations, equivalent intuitionistic fuzzy sets, and (n, m)-rung orthopair fuzzy graphs to enhance performance measures. Moreover, new characterizations of open and closed fuzzy mappings inspired by induced mappings, as well as novel categories of spaces in the framework of generalized fuzzy topologies via fuzzy $g\mu$-closed sets, have further advanced the theoretical landscape, providing a fertile ground for deeper analysis of fuzzy β-connectedness and related concepts.

2. Preliminaries

Definition 2.1. [13] *If X is a collection of objects denoted generically by x, then a fuzzy set A in X is a set of ordered pairs $A = \{(x, \mu_A(x)) : x \in X\}$, where $\mu_A : X \to [0, 1]$ is the membership function (generalized characteristic function) of A.*

Definition 2.2. [13] *Let A and B be fuzzy sets on a non-empty set X. The fundamental operations are defined as follows:*
1. Subset: $A \leq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ for all $x \in X$.
2. Complement: $A^c(x) = 1 - \mu_A(x)$, for all $x \in X$.
3. Union: $(A \vee B)(x) = \max\{\mu_A(x), \mu_B(x)\}$.
4. Intersection: $(A \wedge B)(x) = \min\{\mu_A(x), \mu_B(x)\}$.

These basic operations will be used throughout the paper.

Definition 2.3. [6] *A fuzzy topology on a non-empty set X is a family τ of fuzzy subsets in X satisfying the following axioms :*
1. $0_X, 1_X \in \tau$,
2. $G_1 \wedge G_2 \in \tau$ *for any* $G_1, G_2 \in \tau$,
3. $\vee G_i \in \tau$ *for every* $\{G_i : i \in J\} \leq \tau$.
In this case the pair (X, τ) is called a fuzzy topological space. The elements of τ are called fuzzy open sets. A fuzzy set A is fuzzy closed if and only if A^c is fuzzy open.

Definition 2.4. [6] *Let (X, τ) be fuzzy topological space and let A be a fuzzy set in X. Then, the fuzzy closure and fuzzy interior of A are defined by*
$cl(A) = \wedge\{K : K$ *is a fuzzy closed set in* X *and* $A \leq K\}$
$int(A) = \vee\{G : G$ *is a fuzzy open set in* X *and* $G \leq A\}$.
It can be also shown that $cl(A)$ is a fuzzy closed set and $int(A)$ is a fuzzy open set in X. Further A is fuzzy open set if and only if $A = int(A)$ and A is fuzzy closed set if and only if $A = cl(A)$.

Definition 2.5. [1] *A fuzzy set A in a fuzzy topological space X is called a fuzzy β-open set if $A \leq cl(int(cl(A)))$, and fuzzy β-closed set if $A \geq int(cl(int(A)))$.*

Example 2.1. Let $X = \{a, b, c\}$ and let A,B and C be fuzzy sets in X defined by $A = \{\langle a, 0.1\rangle, \langle b, 0.1\rangle, \langle c, 0.5\rangle\}$, $B = \{\langle a, 1.0\rangle, \langle b, 0.7\rangle, \langle c, 0.1\rangle\}$, and $C = \{\langle a, 0.1\rangle, \langle b, 0.4\rangle, \langle c, 0.5\rangle\}$. Then $\tau = \{0_X, A, B, A \vee B, 1_X\}$ is a fuzzy topology on X and C is a fuzzy β-open set in X.

Definition 2.6. [5] *Let (X, τ) be a fuzzy topological space and let U be a fuzzy set on X. Then, the*

(i) fuzzy β-interior of U, denoted by $\beta int(U)$ is the union of all fuzzy β-open sets of X contained in U. That is, $\beta int(U) = \vee\{G : G$ is a fuzzy β-open set in X and $G \leq U\}$.

(ii) fuzzy β-closure of U, denoted by $\beta cl(U)$, is the intersection of all fuzzy β-closed sets of X contained in U. That is, $\beta cl(U) = \wedge\{G : G$ is a fuzzy β-open set in X and $G \geq U\}$.

Definition 2.7. [5] *A map $f : (X, \tau) \longrightarrow (Y, \sigma)$ is called a M-fuzzy β-continuous map if the inverse image of every fuzzy β-open set in Y is a fuzzy β-open set in X.*

Definition 2.8. [4] *A map $f : (X, \tau) \longrightarrow (Y, \sigma)$ is called a fuzzy β-open map if $f(U)$ is a fuzzy β-open set in Y for every fuzzy open set U in X.*

Definition 2.9. [7] *Let (X, τ) be a fuzzy topological space and let A be a fuzzy* *set in X. If there exist two fuzzy open sets B and C in X such that $B \vee C = A$ and $B \wedge C = 0_X$, then the fuzzy set A is called a fuzzy disconnected set in X. If there does not exist such two fuzzy open sets, then the fuzzy set A is called a fuzzy connected set in X.*

3. Main Results

Definition 3.1. *Let (X, τ) be a fuzzy topological space and let A and B be two non empty fuzzy sets in X. Then, A and B are called fuzzy β-separated sets in X if $A \wedge \beta cl(B) = 0_X$ and $\beta cl(A) \wedge B = 0_X$.*

Proposition 3.1. *Let (X, τ) be a fuzzy topological space and let A and B be two non empty fuzzy sets in X. If A and B are fuzzy β-separated sets and A_1 and B_1 are non empty fuzzy sets in X such that $A_1 \leq A$ and $B_1 \leq B$, then A_1 and B_1 are also fuzzy β-separated sets in X.*

Proof. Let A and B be fuzzy β-separated sets in a fuzzy topological space (X, τ). Then we have $\beta cl(A) \wedge B = 0_X$ and $A \wedge \beta cl(B) = 0_X$. Since $A_1 \leq A$ and $B_1 \leq B$, we get $\beta cl(A_1) \leq \beta cl(A)$ and $\beta cl(B_1) \leq \beta cl(B)$ which implies that $\beta cl(A_1) \wedge B_1 \leq \beta cl(A) \wedge B = 0_X$ and $\beta cl(B_1) \wedge A_1 \leq \beta cl(B) \wedge A = 0_X$. Hence A_1 and B_1 are fuzzy β-separated sets.

Proposition 3.2. *Let (X, τ) be a fuzzy topological space and let A and B be two non empty fuzzy sets in X. If either A and B are both fuzzy β-open sets or fuzzy β-closed sets in X such that $A^c \wedge B^c = 0_X$, then A^c and B^c are fuzzy β-separated sets in X.*

Proof. Let A and B be two non empty fuzzy β-open sets such that $A^c \wedge B^c = 0_X$. Then A^c and B^c are fuzzy β-closed sets. Since $A^c \wedge B^c = 0_X$, we get $\beta cl(A^c) \wedge B^c = 0_X$ and $A^c \wedge \beta cl(B^c) = 0_X$. Hence A^c and B^c are fuzzy β-separated sets.

If A and B are both fuzzy β-closed sets, the result could be obtained by a similar proof.

Theorem 3.1. *Let (X, τ) be a fuzzy topological space and let A and B be two non empty fuzzy sets in X. If either A and B are both fuzzy β-open sets or fuzzy β-closed sets in X and if $H = A \wedge B^c$ and $G = B \wedge A^c$, then H and G are fuzzy β-separated sets in X.*

Proof. Let A and B be both fuzzy β-open sets in X. Then, A^c and B^c are both fuzzy β-closed sets. Since $H \leq B^c$, we get $\beta cl(H) \leq \beta cl(B^c) = B^c$, which implies that $\beta cl(H) \wedge B \leq B^c \wedge B = 0_X$. Thus $\beta cl(H) \wedge B = 0_X$. Since $G \leq B$, we get $\beta cl(H) \wedge G \leq \beta cl(H) \wedge B = 0_X$. Thus $\beta cl(H) \wedge G = 0_X$. Similarly we can get $H \wedge \beta cl(G) = 0_X$. Hence H and G are fuzzy β-separated sets.

If A and B are both fuzzy β-closed sets, the result could be obtained by a similar proof.

Theorem 3.2. *Let(X,τ) be a fuzzy topological space and let A,B and C be non empty fuzzy sets in X. If B and C are fuzzy β-separated sets in X, then $A \wedge B$ and $A \wedge C$ are fuzzy β-separated sets in X.*
Proof. Let B and C be fuzzy β-separated sets in X. Then, we have $B \wedge \beta cl(C) = 0_X$ and $\beta cl(B) \wedge C = 0_X$. Since $A \wedge B \leq B$ and $A \wedge C \leq C$, we get $\beta cl(A \wedge B) \leq \beta cl(B)$ and $\beta cl(A \wedge C) \leq \beta cl(C)$. Thus $\beta cl(A \wedge B) \wedge (A \wedge C) \leq \beta cl(B) \wedge C = 0_X$. So $\beta cl(A \wedge B) \wedge (A \wedge C) = 0_X$. Similarly we can get $(A \wedge B) \wedge \beta cl(A \wedge C) = 0_X$. Hence $A \wedge B$ and $A \wedge C$ are fuzzy β-separated sets.

Theorem 3.3. *Let (X,τ) be a fuzzy topological space and let A and B be two non empty fuzzy sets in X. Then, A and B are fuzzy β-separated sets if and only if there exists fuzzy β-open sets A_1 and B_1 in the fuzzy topological space (X,τ) such that $A \leq A_1$, $B \leq B_1$ and $A \wedge B_1 = 0_X$, $A_1 \wedge B = 0_X$.*
Proof. Let A and B be two fuzzy β-separated sets in a fuzzy topological space (X,τ). Then, $A \wedge \beta cl(B) = 0_X$ and $\beta cl(A) \wedge B = 0_X$. Take $A_1 = (\beta cl(B))^c$ and $B_1 = (\beta cl(A))^c$. Then, A_1 and B_1 are fuzzy β-open sets in X such that $A \leq A_1$, $B \leq B_1$. Since $A_1 = (\beta cl(B))^c$ and $B_1 = (\beta cl(A))^c$, we get $A_1 \wedge \beta cl(B) = (\beta cl(B))^c \wedge \beta cl(B) = 0_X$ and $\beta cl(A) \wedge B_1 = \beta cl(A) \wedge (\beta cl(A))^c = 0_X$. Thus $A_1 \wedge \beta cl(B) = 0_X$ and $\beta cl(A) \wedge B_1 = 0_X$. This implies that $A_1 \wedge B = 0_X$ and $A \wedge B_1 = 0_X$.
Conversely assume that A_1 and B_1 be fuzzy β-open sets such that $A \leq A_1$, $B \leq B_1$, $A \wedge B_1 = 0_X$, and $B \wedge A_1 = 0_X$. Then $A \leq B_1^c$, $B \leq A_1^c$ and A_1^c and B_1^c are fuzzy β-closed sets. Therefore we get $\beta cl(A) \leq \beta cl(B_1^c) = B_1^c$ and $\beta cl(B) \leq \beta cl(A_1^c) = A_1^c$. Thus $\beta cl(A) \leq B_1^c$ and $\beta cl(B) \leq A_1^c$. Since $A_1^c \leq A^c$ and $B_1^c \leq B^c$, we get $\beta cl(A) \leq B_1^c \leq B^c$ and $\beta cl(B) \leq A_1^c \leq A^c$. That is $\beta cl(A) \leq B^c$ and $\beta cl(B) \leq A^c$. Therefore $\beta cl(A) \wedge B \leq B^c \wedge B = 0_X$ and $\beta cl(B) \wedge A \leq A^c \wedge A = 0_X$. Thus $\beta cl(A) \wedge B = 0_X$ and $\beta cl(B) \wedge A = 0_X$. Hence A and B are fuzzy β-separated sets.

Definition 3.2. *Let (X,τ) be a fuzzy topological space and let A be a fuzzy set in X. If there exist two fuzzy β-open sets B and C such that $B \vee C = A$ and $B \wedge C = 0_X$, then the fuzzy set A is called a fuzzy β-disconnected set in X. If there does not exist such two fuzzy β-open sets, then the fuzzy set A is called a fuzzy β-connected set in X.*

Example 3.1. Let $X = \{a, b\}$ with $\tau = \{0_X, A, B, C, 1_X\}$, where $A = \{\langle a, 1\rangle, \langle b, 0.7\rangle\}$, $B = \{\langle a, 0\rangle, \langle b, 0.7\rangle\}$, and $C = \{\langle a, 1\rangle, \langle b, 0\rangle\}$ are fuzzy open sets in X. Since every fuzzy open set is a fuzzy β-open set, A,B and C are fuzzy β-open sets in X. Now $B \wedge C = \{\langle a, 0\rangle, \langle b, 0\rangle\} = 0_X$ and $B \vee C = \{\langle a, 1\rangle, \langle b, 0.7\rangle\} = A$. Hence

A is a fuzzy β-disconnected set.

Definition 3.3. *A fuzzy topological space X is said to be a fuzzy β-disconnected space if there exist non empty fuzzy β-open sets A and B in X such that $A \vee B = 1_X$ and $A \wedge B = 0_X$. If X is not a fuzzy β-disconnected space, then it is said to be a fuzzy β-connected space.*

Example 3.2. Let $X = \{a, b, c\}$ with $\tau = \{0_X, A, B, 1_X\}$ where $A = \{\langle a, 1\rangle, \langle b, 0\rangle, \langle c, 0\rangle\}$, and $B = \{\langle a, 0\rangle, \langle b, 1\rangle, \langle c, 1\rangle\}$ are fuzzy open sets in X. Since every fuzzy open set is a fuzzy β-open set, we get A and B are fuzzy β-open sets in X. Now $A \neq 0_X$, $B \neq 0_X$, $A \vee B = \{\langle a, 1\rangle, \langle b, 1\rangle, \langle c, 1\rangle\} = 1_X$ and $A \wedge B = \{\langle a, 0\rangle, \langle b, 0\rangle, \langle c, 0\rangle\} = 0_X$. Thus X is a fuzzy β-disconnected space.

Example 3.3. Let $X = \mathbb{N} = \{1, 2, 3, \dots\}$ with $\tau = \{0_X, A, B, 1_X\}$ where $A = \{\langle n, 1\rangle : n \text{ is even}\} \vee \{\langle n, 0\rangle : n \text{ is odd}\}$ and $B = \{\langle n, 1\rangle : n \text{ is odd}\} \vee \{\langle n, 0\rangle : n \text{ is even}\}$ are fuzzy open sets in X. Since every fuzzy open set is a fuzzy β-open set, we get A and B are fuzzy β-open sets in X. Now $A \neq 0_X$, $B \neq 0_X$, $A \vee B = \{\langle n, 1\rangle : n \in \mathbb{N}\} = 1_X$ and $A \wedge B = \{\langle n, 0\rangle : n \in \mathbb{N}\} = 0_X$. Thus X is a fuzzy β-disconnected space.

Lemma 3.1. *Every fuzzy β-connected set in a fuzzy topological space (X, τ) is connected set in (X, τ) .*

Proof. Let A be a fuzzy β-connected set in a fuzzy topological space (X, τ). Suppose that A is disconnected set in (X, τ). Then, there exist two fuzzy open sets B and C such that $B \vee C = A$ and $B \wedge C = 0_X$. But B and C are also being fuzzy β-open sets. Thus A is fuzzy β-disconnected set in X, which is a contradiction.

Theorem 3.4. *Let (X, τ) and (Y, σ) be two fuzzy topological spaces and let $f : X \longrightarrow Y$ be a one to one M-fuzzy β-continuous map. If A is fuzzy β-connected in X, then $f(A)$ is fuzzy β-connected in Y.*

Proof. Let A be fuzzy β-connected set in (X, τ). Suppose that $f(A)$ is fuzzy β-disconnected set. Then, there exist two fuzzy β-open sets B and C in Y such that $B \vee C = f(A)$, and $B \wedge C = 0_Y$. Since f is one to one map, $A = f^{-1}(f(A)) = f^{-1}(B \vee C) = f^{-1}(B) \vee f^{-1}(C)$ and $f^{-1}(B) \wedge f^{-1}(C) = f^{-1}(B \wedge C) = f^{-1}(0_Y) = 0_X$. That is, $A = f^{-1}(B) \vee f^{-1}(C)$ and $f^{-1}(B) \wedge f^{-1}(C) = 0_X$. Since f is M-fuzzy β-continuous map, $f^{-1}(B)$ and $f^{-1}(C)$ are fuzzy β-open sets in X. This contradict the fact that A is fuzzy β-connected, hence $f(A)$ is a fuzzy β-connected set.

Theorem 3.5. *Let X and Y be two fuzzy topological spaces and let $f : X \to Y$ be a surjective M-fuzzy β-continuous map. If X is a fuzzy β-connected space, then Y is a fuzzy β-connected space.*

Proof. Let X be a fuzzy β-connected space. Suppose that Y is a fuzzy β-

disconnected space. Then there exist two non empty fuzzy β-open sets B and C such that $B \vee C = 1_Y$ and $B \wedge C = 0_Y$. Thus $1_X = f^{-1}(1_Y) = f^{-1}(B \vee C) = f^{-1}(B) \vee f^{-1}(C)$ and $0_X = f^{-1}(0_Y) = f^{-1}(B \wedge C) = f^{-1}(B) \wedge f^{-1}(C)$. That is $f^{-1}(B) \vee f^{-1}(C) = 1_X$ and $f^{-1}(B) \wedge f^{-1}(C) = 0_X$. Since f is M-fuzzy β-continuous map, $f^{-1}(B)$ and $f^{-1}(C)$ are non empty fuzzy β-open sets in X. This contradict the fact that X is a fuzzy β-connected space, hence Y is a fuzzy β-connected space.

Proposition 3.3. *The only fuzzy sets in X which are both fuzzy β-open and fuzzy β-closed are 0_X or 1_X if and only if X is a fuzzy β-connected space.*

Proof. Suppose that the only fuzzy sets in X which are both fuzzy β-open and fuzzy β-closed are 0_X and 1_X. Let X be fuzzy β-disconnected space. Then, there exist two non empty fuzzy β-open sets A and B such that $A \vee B = 1_X$ and $A \wedge B = 0_X$. Since $A = B^c$, A is fuzzy β-closed. By our assumption, $A = 0_X$ or $A = 1_X$ which is a contradiction to that A is non empty fuzzy β-open set. Therefore X is fuzzy β-connected space.

Conversely suppose that X is a fuzzy β-connected space. Let A be a non empty fuzzy set in X which is both fuzzy β-open and fuzzy β-closed. Then A and A^c are fuzzy β-open sets such that $A \wedge A^c = 0_X$ and $A \vee A^c = 1_X$. Since X is fuzzy β-connected space, either $A = 0_X$ or $A^c = 0_X$. That is either $A = 0_X$ or $A = 1_X$.

Theorem 3.6. *Let (X, τ) and (Y, σ) be two fuzzy topological spaces and let $f : X \longrightarrow Y$ a one to one map which is both fuzzy β-open and fuzzy β-closed map. If Y is a fuzzy β-connected space, then X is a fuzzy β-connected space.*

Proof. Let A be both fuzzy open and fuzzy closed set in X and let f be one to one which is both fuzzy β-open and fuzzy β-closed map. Then, $f(A)$ is both fuzzy β-open and fuzzy β-closed set in Y. Since Y is fuzzy β-connected space, $f(A) = 0_Y$ or $f(A) = 1_Y$. Since f is one to one, we get $A = 0_X$ or $A = 1_X$. Hence X is fuzzy β-connected space.

Theorem 3.7. ıIf a fuzzy set A of a fuzzy β-connected space X is not the union of any two fuzzy β-separated sets, then A is a fuzzy β-connected set.

Proof. Let X be a fuzzy β-connected space and let A be a fuzzy set in X which is not the union of any two fuzzy β-separated sets. Assume that A is fuzzy β-disconnected set. Then, $A = H \vee K$, where H and K are non empty disjoint fuzzy β-open sets in X. Since $H \leq K^c$ and $K \leq H^c$, we get $\beta cl(H) \leq \beta cl(K^c) = K^c$ and $\beta cl(K) \leq \beta cl(H^c) = H^c$. This implies that $\beta cl(H) \wedge K \leq K^c \wedge K = 0_X$ and $H \wedge \beta cl(K) \leq H \wedge H^c = 0_X$.Thus $\beta cl(H) \wedge K = 0_X$ and $H \wedge \beta cl(K) = 0_X$. Thus H and K are fuzzy β-separated sets. This is a contradiction, hence A is fuzzy β-connected set.

Remark 3.1. *It is essential to discuss whether all the classical properties of β-connectedness are satisfied within the framework of fuzzy β-connectedness. Some classical properties may not hold in this generalized fuzzy setting, and counterexamples can be provided to illustrate such cases.*

Example 3.4. Let $X = \{a, b, c\}$ with fuzzy topology $\tau = \{0_X, A, B, 1_X\}$, where $A = \{\langle a, 1\rangle, \langle b, 0\rangle, \langle c, 0\rangle\}$ and $B = \{\langle a, 0\rangle, \langle b, 1\rangle, \langle c, 0\rangle\}$ are fuzzy β-open sets in X. Now $A \wedge B = \{\langle a, 0\rangle, \langle b, 0\rangle, \langle c, 0\rangle\} = 0_X$ and $A \vee B = \{\langle a, 1\rangle, \langle b, 1\rangle, \langle c, 0\rangle\} \neq 1_X$. Hence X is not fuzzy β-connected, illustrating that some classical properties of connectedness may not hold in the fuzzy β-connected framework.

4. Conclusion

In this paper, we have introduced and systematically investigated the concept of fuzzy β-connectedness in fuzzy topological spaces. By extending classical and fuzzy notions of connectedness, we developed a generalized framework based on fuzzy β-open sets and fuzzy β-separated sets, and provided illustrative examples to demonstrate how some classical properties of connectedness may not hold in this generalized fuzzy setting. Several theorems were established to characterize fuzzy β-connected and fuzzy β-disconnected sets, as well as their preservation under M-fuzzy β-continuous and fuzzy β-open mappings. Our study also identified conditions under which fuzzy sets are both fuzzy β-open and fuzzy β-closed, providing insight into the structural behavior of fuzzy β-connected spaces. These results enrich the theoretical understanding of fuzzy connectedness and lay a foundation for future investigations, including its relationships with other generalized fuzzy open sets, compactness, convergence, and applications in modeling uncertainty in decision-making processes.

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