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ANALYSIS OF NON-SIMULTANEOUS NUMERICAL BLOW-UP IN SYSTEMS OF HEAT EQUATIONS WITH n COMPONENTS AND NONLINEAR BOUNDARY CONDITIONS

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Abstract: This paper concerns the study of a numerical approximation for a system of heat equations with n components and nonlinear boundary conditions. We show that the solution of the semidiscrete problem, obtained by the finite difference method, blows up in finite time. We also establish conditions under which non-simultaneous or simultaneous blow-up occurs for the semidiscrete problem. After proving the convergence of the numerical blow-up time, we conclude by presenting numerical results that illustrate key aspects of our study.

Keywords and Phrases: System of heat equations, n components, semidiscretization, non-simultaneous blow-up, simultaneous blow-up, convergence, numerical blow-up time, arc-length transformation, Aitken's Δ^2 method.

2020 Mathematics Subject Classification: Primary 35K51, 35B55, Secondary 65M06, 65M12.

1. Introduction

In this paper, we consider the following system of heat equations with n components and nonlinear boundary conditions:

$$\begin{cases} (u_j)_t(x,t) = (u_j)_{xx}(x,t), & (x,t) \in (-1,1) \times (0,T), \\ -(u_j)_x(-1,t) = (u_j^{p_j} u_{j+1}^{q_{j+1}})(-1,t), & t \in (0,T), \\ (u_j)_x(1,t) = (u_j^{p_j} u_{j+1}^{q_{j+1}})(1,t), & t \in (0,T), \\ u_j(x,0) = u_{j,0}(x), & x \in [-1,1], \quad j = 1,\dots,n, \quad n \geq 2, \\ u_{n+1} = u_1, \quad p_{n+1} = p_1, \quad q_{n+1} = q_1, & \end{cases} \tag{1}$$

where the constants $p_j, q_j \geq 0$ for $j = 1,\dots,n$ and the initial data $u_{j,0}$, $j = 1,\dots,n$, are positive, smooth, even functions satisfying the compatibility conditions.

This type of multi-component heat equation system with nonlinear boundary conditions arises in various applied contexts. For instance, it can model autocatalytic chemical reactions in tubular reactors, where nonlinear loss terms at the boundaries correspond to reactive walls [23, 17]. In heat transfer, it can describe conduction in a rod subject to nonlinear radiative cooling (e.g., Stefan-Boltzmann law) at the ends, possibly coupled with heat exchange between components [7, 2]. In biology, it may represent the spatial diffusion of multiple interacting species, where exchanges or reactions occur at the boundaries, such as nutrient absorption or toxin release through reactive membranes [18, 20]. More generally, such systems also appear in nonlinear diffusion processes with coupled boundary dynamics, including transport across semi-permeable or reactive interfaces in industrial or environmental contexts [6, 10].

Previous studies have shown the existence and uniqueness of local classical solution $(u_1,\dots,u_n)$ to system (1) (see, for instance, [14]). Here, $[0,T)$ denotes the maximal time interval on which the solution exists. The time T may be either finite or infinite. If $T = +\infty$, the solution is said to exist globally. If $T < +\infty$, then the solution develops a singularity in finite time, that is,

$$\limsup_{t \to T} \sum_{j=1}^{n} \|u_j(\cdot,t)\|_\infty = +\infty,$$

where $\|u_j(\cdot,t)\|_\infty = \max_{-1 \leqslant x \leqslant 1} |u_j(x,t)|$, for $j = 1,\dots,n$.

In this case, we say that the solution $(u_1,\dots,u_n)$ blows up in finite time, and T is called the blow-up time.

Simultaneous and non-simultaneous blow-up phenomena for systems with nonlinear boundary conditions have attracted much attention (see, e.g., [3, 4, 5, 15, 21]). We say that simultaneous blow-up occurs if all components of the solution blow up at the same time while non-simultaneous blow-up means that at least $j \in \{1, \ldots, n\}$ components blow up while the others remain bounded up to the blow-up time.

In [16], the authors theoretically studied various blow-up scenarios for system (1) in a domain $B_R \subset \mathbb{R}^N$, including the cases where:

- Only one component blows up;
- Exactly two components blow up;
- Blow-up may be either simultaneous or non-simultaneous, for all initial data.

In particular, they proved that for all initial data, non-simultaneous blow-up occurs in finite time (for fixed $j \in \{1, \ldots, n\}$):

- If $k \in \{0, 1, \ldots, n-2\}$, $\beta_\eta = \dfrac{1 - q_{\eta+1}\beta_{\eta+1}}{p_\eta - 1} > 0, \quad p_\eta < 1 \quad (\eta = j-1, j-2, \ldots, j-k), q_{j-k}\beta_{j-k} < 1$, with $\beta_j = \frac{1}{p_j - 1}, \quad p_m \leq 1 < p_j \ (m = 1, \ldots, j-1, j+1, \ldots, n)$.

Simultaneous blow-up occurs in finite time (for fixed $j \in \{1, \ldots, n\}$):

- If $k = n-1$, $\beta_\eta = \dfrac{1 - q_{\eta+1}\beta_{\eta+1}}{p_\eta - 1} > 0$, $p_\eta < 1 \quad (\eta = j-1, j-2, \ldots, j+1-n), \quad \beta_\eta > 0 (\eta = j-1, j-2, \ldots, j+2-n), \quad \beta_{j+1-n} \geq 0$, with $\beta_j = \dfrac{1}{p_j - 1}, \quad p_m \leq 1 < p_j \ (m = 1, 2, \ldots, j-1, j+1, \ldots, n)$.

This work aims to study the numerical approximation of system (1) using the finite difference method, under the blow-up conditions described above, with particular attention paid to the estimation of the blow-up time.

In this context, the numerical approximation of coupled parabolic systems exhibiting blow-up continues to attract considerable interest, as evidenced by several recent works devoted to finite difference schemes [11, 12, 13].

Our study is in line with the works [8, 9] on the numerical approximation of nonlinear parabolic systems, as well as the references cited therein.

We organize this paper as follows. In Section 2, we present a semidiscrete scheme for problem (1). Section 3 contains some properties of this scheme. In Section

4, we prove that the solution of the semidiscrete scheme blows up in finite time under certain conditions. Section 5 proposes criteria to distinguish between non-simultaneous and simultaneous blow-up. In Section 6, we prove the convergence of the solution and the numerical blow-up times as the mesh size tends to zero. Section 7 presents some numerical experiments, including a discussion of the results. Finally, Section 8 concludes the paper.

2. Semidiscrete problem

Let $I \geq 2$ be a positive integer and define the grid $x_i = -1 + (i-1)h$, $i = 1, \ldots, I$, where $h = \dfrac{2}{I-1}$ is the mesh parameter. We approximate the solution $(u_1, \ldots, u_n)$ of the problem (1) by $(U_{1,h}, \ldots, U_{n,h})$ and approximate the initial data $(u_{1,0}, \ldots, u_{n,0})$ of the same problem by $(\varphi_{1,h}, \ldots, \varphi_{n,h})$. By the finite difference method, it is easy to see that $(U_{1,h}, \ldots, U_{n,h}) \in \big(C^1([0,T_h), \mathbb{R}^I)\big)^n$ is a solution of the following ODEs system:

$$U'_{j,i}(t) = \delta^2 U_{j,i}(t) + \Upsilon_{j,i} U_{j,i}^{p_j}(t) U_{j+1,i}^{q_{j+1}}(t), \quad i = 1, \ldots, I, \quad t \in [0, T_h), \tag{2}$$

$$U_{j,i}(0) = \varphi_{j,i}, \quad i = 1, \ldots, I, \quad j = 1, 2, \ldots, n, \quad n \geq 2, \tag{3}$$

$$U_{n+1,i} = U_{1,i}, \quad p_{n+1} = p_1, \quad q_{n+1} = q_1, \quad i = 1, \ldots, I, \tag{4}$$

where

$$\varphi_{j,i} > 0, \quad \varphi_{j,I+1-i} = \varphi_{j,i}, \quad 1 \leqslant i \leqslant I, \quad p_j, q_j \geq 0,$$

$$\begin{aligned} \delta^2 U_{j,i}(t) &= \frac{U_{j,i+1}(t) - 2U_{j,i}(t) + U_{j,i-1}(t)}{h^2}, \quad 2 \leqslant i \leqslant I-1, \quad t \in [0, T_h), \\ \delta^2 U_{j,1}(t) &= \frac{2U_{j,2}(t) - 2U_{j,1}(t)}{h^2}, \quad t \in [0, T_h), \\ \delta^2 U_{j,I}(t) &= \frac{2U_{j,I-1}(t) - 2U_{j,I}(t)}{h^2}, \quad t \in [0, T_h), \end{aligned}$$

$$\Upsilon_{j,1} = \Upsilon_{j,I} = \frac{2}{h}, \quad \Upsilon_{j,i} = 0, \quad 2 \leqslant i \leqslant I-1.$$

Here, $U_{j,h}(t) = (U_{j,1}(t), \ldots, U_{j,I}(t))^T$, $\varphi_{j,h} = (\varphi_{j,1}, \ldots, \varphi_{j,I})^T$, and $[0, T_h)$ is the maximal time interval on which

$$\max\{\|U_{1,h}(t)\|_\infty, \ldots, \|U_{n,h}(t)\|_\infty\} < \infty,$$

where

$$\|U_{j,h}(t)\|_\infty = \max_{1\leqslant i\leqslant I} |U_{j,i}(t)|\,, \quad j = 1, 2, \ldots, n.$$

When the time T_h is finite, we say that the solution $(U_{1,h}, \ldots, U_{n,h})$ blows up in finite time, and the time T_h is called the blow-up time.

3. Properties of the semidiscrete scheme

In this section, we proceed with methods similar to those of [8, 9, 19] to present some auxiliary results for problem (2)–(4), without proof.

Definition 1. *We say that* $(\underline{U}_{j,h})_{j=1}^n$ *is a lower solution of* (2)–(4)*, where* $\underline{U}_{j,h} \in C^1([0,T_h), \mathbb{R}^I)$ *for* $j = 1, 2, \ldots, n$*, if*

$$\underline{U'}_{j,i}(t) \leqslant \delta^2 \underline{U}_{j,i}(t) + \Upsilon_{j,i} \underline{U}_{j,i}^{p_j}(t) \underline{U}_{j+1,i}^{q_{j+1}}(t), \quad i = 1, \ldots, I, \; t \in (0, T_h),$$

$$\underline{U}_{j,i}(0) \leqslant \varphi_{j,i}, \quad i = 1, \ldots, I.$$

Similarly, $(\overline{U}_{j,h})_{j=1}^n$ *is called an upper solution of* (2)–(4)*, where* $\overline{U}_{j,h} \in C^1([0,T_h), \mathbb{R}^I)$ *for* $j = 1, 2, \ldots, n$*, if the inequalities are reversed.*

Lemma 1. (Discrete maximum principle) *Let* $\alpha_{j,h}, \beta_{j,h} \in C^0([0,T_h), \mathbb{R}^I)$ *and* $U_{j,h} \in C^1([0,T_h), \mathbb{R}^I)$ *such that*

$$U'_{j,i}(t) - \delta^2 U_{j,i}(t) - \alpha_{j,i}(t) U_{j,i}(t) - \beta_{j,i}(t) U_{j+1,i}(t) \geqslant 0, \quad i = 1, \ldots, I, \; t \in (0, T_h),$$
$$U_{j,i}(0) \geqslant 0, \quad i = 1, \ldots, I, \quad j = 1, 2, \ldots, n,$$
$$U_{n+1,i}(t) = U_{1,i}(t), \quad i = 1, \ldots, I.$$

Then we have

$$U_{j,i}(t) \geqslant 0, \quad i = 1, \ldots, I, \quad j = 1, 2, \ldots, n, \quad t \in (0, T_h).$$

Lemma 2. (Comparison principle) *Let* $(\underline{U}_{j,h})_{j=1}^n$ *and* $(\overline{U}_{j,h})_{j=1}^n$ *be, respectively, lower and upper solutions of* (2)–(4)*, where* $\underline{U}_{j,h}, \overline{U}_{j,h} \in C^1([0,T_h), \mathbb{R}^I)$ *for* $j = 1, 2, \ldots, n$*, and assume that* $\underline{U}_{j,h}(0) \leqslant \overline{U}_{j,h}(0), \quad j = 1, 2, \ldots, n$*. Then*

$$\underline{U}_{j,h} \leqslant \overline{U}_{j,h}, \quad j = 1, 2, \ldots, n.$$

Lemma 3. (Further properties) *Let* $k = \lfloor (I+1)/2 \rfloor$*, where* $\lfloor \cdot \rfloor$ *denotes the integer part and let* $(U_{j,h})_{j=1}^n$ *be the solution of (2)–(4), where* $U_{j,h} \in C^1([0,T_h), \mathbb{R}^I)$ *for* $j = 1, 2, \ldots, n$*, with initial data* $(\varphi_{j,h})_{j=1}^n$ *such that* $0 < \varphi_{j,i} < \varphi_{j,i+1}, \quad j = 1, 2, \ldots, n, \quad i = k, \ldots, I-1$*. Then we have*

(i) $U_{j,h}(t) \geq \varphi_{j,h}$, $j = 1, 2, \ldots, n$, $t \in (0, T_h)$;

(ii) $U_{j,I+1-i} = U_{j,i}$, $i = 1, \ldots, I$;

(iii) $U_{j,i+1}(t) > U_{j,i}(t)$, $j = 1, 2, \ldots, n$, $i = k, \ldots, I-1$, $t \in (0, T_h)$;

(iv) $U'_{j,i}(t) \geq 0$, $j = 1, 2, \ldots, n$, $i = k, \ldots, I$, $t \in (0, T_h)$.

4. Blow-up of the semidiscrete solution

In this section, under certain assumptions, we provide conditions for the global existence of the solution of the semidiscrete problem, and we also show that the solution $(U_{1,h}, \ldots, U_{n,h})$ of (2)–(4) blows up in finite time. We characterize the blow-up or global existence of the solution $(U_{1,h}, \ldots, U_{n,h})$ of (2)–(4) in terms of the matrix A, defined as follows:

$$A = \begin{pmatrix} p_1 & q_2 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & p_2 & q_3 & 0 & & & \vdots \\ 0 & 0 & p_3 & q_4 & 0 & & \\ \vdots & 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & 0 & 0 & p_{n-1} & q_n \\ q_1 & \ldots & \ldots & \ldots & 0 & 0 & p_n \end{pmatrix}.$$

For convenience, we define $p_{n+l} = p_l$ and $q_{n+l} = q_l$ for all integers l.
Let $X = (\alpha_1, \ldots, \alpha_n)$ be the solution of

$$(A - Id)\, X^T = (-1, \ldots, -1)^T. \tag{5}$$

It is easy to see that α_j is a fraction whose denominator is $\prod_{k=1}^{n} q_k - \prod_{k=1}^{n}(1-p_k)$ and whose numerator is negative whenever $0 \leq p_j \leq 1$ and $q_j \geq 0$ for $j = 1, 2, \ldots, n$.

Definition 2. *We say that the solution $(U_{1,h}, \ldots, U_{n,h})$ of (2)–(4) blows up in finite time if there exists a finite time $T_h > 0$ such that for $t \in [0, T_h)$, $\max\{\|U_{1,h}(t)\|_\infty, \ldots, \|U_{n,h}(t)\|_\infty\} < \infty$ and*

$$\limsup_{t \to T_h} \sum_{j=1}^{n} \|U_{j,h}(t)\|_\infty = +\infty.$$

The time T_h is called the blow-up time of the solution.

Theorem 1. *If $p_1, \ldots, p_n > 1$, then the solution $(U_{1,h}, \ldots, U_{n,h})$ of (2)–(4) blows* *up in finite time T_h.*

Proof. Assume, by contradiction, that the solution exists for all time $t \geq 0$. For each $j = 1, \ldots, n$, from (2), we have:

$$U'_{j,I}(t) = \frac{2U_{j,I-1}(t) - 2U_{j,I}(t)}{h^2} + \frac{2}{h} U_{j,I}^{p_j}(t) U_{j+1,I}^{q_{j+1}}(t), \quad t \geq 0,$$

with periodic conditions $U_{n+1,I} = U_{1,I}$ and $q_{n+1} = q_1$.
By Lemma 3, there exists a constant $m > 0$ such that $U_{j+1,I}(t) \geq m$ for all $t \geq 0$. Moreover, $U_{j,I-1}(t) < U_{j,I}(t)$ and $U'_{j,I}(t) \geq 0$, so $U_{j,I}(t)$ is increasing. Thus, we have:

$$U'_{j,I}(t) \geq \frac{2U_{j,I-1}(t) - 2U_{j,I}(t)}{h^2} + \frac{2m^{q_{j+1}}}{h} U_{j,I}^{p_j}(t).$$

The diffusive term $\frac{2U_{j,I-1}(t) - 2U_{j,I}(t)}{h^2}$ is negative. However, since $p_j > 1$, the nonlinear term $\frac{2m^{q_{j+1}}}{h} U_{j,I}^{p_j}(t)$ grows faster than the diffusive term when $U_{j,I}(t)$ becomes large. Therefore, there exists a time $t_0 \geq 0$ such that for all $t \geq t_0$:

$$\frac{2m^{q_{j+1}}}{h} U_{j,I}^{p_j}(t) \geq 2 \left| \frac{2U_{j,I-1}(t) - 2U_{j,I}(t)}{h^2} \right|.$$

Consequently, for $t \geq t_0$:

$$U'_{j,I}(t) \geq \frac{2m^{q_{j+1}}}{h} U_{j,I}^{p_j}(t) + \frac{2U_{j,I-1}(t) - 2U_{j,I}(t)}{h^2} \geq \frac{m^{q_{j+1}}}{h} U_{j,I}^{p_j}(t).$$

Define $\beta_j = \frac{h}{m^{q_{j+1}}}$. Then for $t \geq t_0$:

$$U'_{j,I}(t) \geq \frac{1}{\beta_j} U_{j,I}^{p_j}(t). \tag{6}$$

Integrate this inequality from t_0 to t (for $t > t_0$):

$$\int_{U_{j,I}(t_0)}^{U_{j,I}(t)} \frac{dU}{U^{p_j}} \geq \frac{1}{\beta_j} \int_{t_0}^{t} d\tau.$$

Computing the integral yields:

$$\frac{1}{p_j - 1} \left(U_{j,I}^{1-p_j}(t_0) - U_{j,I}^{1-p_j}(t) \right) \geq \frac{t - t_0}{\beta_j}.$$

Under the assumption that the solution exists for all $t \geq 0$, $U_{j,I}(t)$ must remain bounded (otherwise, blow-up would occur at finite time, contradicting global existence). However, the left-hand side of the inequality is bounded (since $U_{j,I}(t)$ is bounded and $p_j > 1$), while the right-hand side grows without bound as $t \to \infty$. This is a contradiction.
Therefore, the solution cannot exist for all time, and blow-up must occur at finite time T_h.
Moreover, by integrating (6) from t_0 to T_h, we obtain an upper bound for T_h:

$$T_h \leq t_0 + \frac{\beta_j}{p_j - 1} U_{j,I}^{1-p_j}(t_0), \quad j = 1, \ldots, n.$$

Thus, if $p_1, \ldots, p_n > 1$, the solution $(U_{1,h}, \ldots, U_{n,h})$ blows up in finite time T_h with

$$T_h \leq \max_{1 \leq j \leq n} \left(t_0 + \frac{\beta_j}{p_j - 1} U_{j,I}^{1-p_j}(t_0) \right).$$

Theorem 2. *Assume that* $0 \leq p_j \leq 1$ *for* $j = 1, 2, \ldots, n$. *Let* $(\alpha_1, \ldots, \alpha_n)$ *be the solution of* (5). *Then :*

(1) *if* $\min\limits_{1 \leq j \leq n} \alpha_j > 0$, *then the solution* $(U_{1,h}, \ldots, U_{n,h})$ *of* (2)–(4) *exists globally.*

(2) *if* $\min\limits_{1 \leq j \leq n} \alpha_j \leq 0$, *then the solution* $(U_{1,h}, \ldots, U_{n,h})$ *of* (2)–(4) *blows up in finite time* T_h.

The following lemma describes the behavior of the positive solutions of

$$\begin{cases} Y_j'(z) = Y_j^{p_j}(z) Y_{j+1}^{q_{j+1}}(z), \\ Y_j(0) = Y_{j,0} > 0, \quad j = 1, 2, \ldots, n, \\ Y_{n+1} = Y_1, \qquad q_{n+1} = q_1, \end{cases} \tag{7}$$

where $0 \leqslant p_j \leqslant 1$ and $q_j \geq 0$.

Lemma 4. *Let* $\{Y_j(z)\}$ *be a positive solution of* (7) *with* A *nonsingular, and let* $(\alpha_1, \ldots, \alpha_n)$ *be the solution of* (5). *Then :*

(i) *If* $\min\limits_{1 \leq j \leq n} \alpha_j > 0$, *then* (7) *admits a global upper solution of the form* $Y_j(z) = L_j(z + z_0)^{\alpha_j}$, *where* $L_j > 0$ *is a constant.*

(ii) *If* $\min\limits_{1 \leq j \leq n} \alpha_j \leq 0$, *then all positive solution of* (7) *blows up.*

Proof. See [22], Theorem 2.1.

Proof of Theorem 2. The proof relies on the construction of lower solutions and upper solutions based on the ODE system (7) and Lemma 4.

Case (1): Global existence when $\min_{1\le j\le n} \alpha_j > 0$

By Lemma 4, there exists a global solution $\phi_j(s) = L_j(s+s_0)^{\alpha_j}$ of system (7) with $L_j > 0$ and $s_0 > 0$ chosen such that $\phi_j(0) \ge \max_{1\le i\le I} \varphi_{j,i}$. Let $b(t)$ be a strictly increasing, positive, and continuous function such that $b'(t) \ge \max_{j,i} \Upsilon_{j,i}$. Define the upper-solution:

$$\overline{U}_{j,i}(t) = \phi_j(b(t)), \quad j = 1,\dots,n,\ i = 1,\dots,I.$$

Then:

$$\overline{U}'_{j,i}(t) = b'(t)\phi'_j(b(t)) \ge b'(t)\phi_j^{p_j}(b(t))\phi_{j+1}^{q_{j+1}}(b(t)) \ge \Upsilon_{j,i}\overline{U}^{p_j}_{j,i}(t)\overline{U}^{q_{j+1}}_{j+1,i}(t),$$

and $\overline{U}_{j,i}(0) = \phi_j(b(0)) \ge \phi_j(0) \ge \varphi_{j,i}$. Since $\overline{U}_{j,i}(t)$ is constant in space, $\delta^2\overline{U}_{j,i}(t) = 0$. Thus:

$$\overline{U}'_{j,i}(t) \ge \delta^2\overline{U}_{j,i}(t) + \Upsilon_{j,i}\overline{U}^{p_j}_{j,i}(t)\overline{U}^{q_{j+1}}_{j+1,i}(t).$$

By the comparison principle (Lemma 2), $(U_{1,h},\dots,U_{n,h})$ exists globally.

Case (2): Finite-time blow-up when $\min_{1\le j\le n} \alpha_j \le 0$

By Lemma 4, any positive solution of system (7) blows up in finite time. Choose $\kappa > 0$ sufficiently small and $\varepsilon > 0$ such that $\varepsilon \le 2\kappa \le \dfrac{2}{h}$. Let $a_i = \kappa x_i^2$ for $i = 1,\dots,I$, where $x_i = -1 + (i-1)h$.
Define the lower solution:

$$\underline{U}_{j,i}(t) = \phi_j(a_i + \varepsilon t), \quad j = 1,\dots,n,\ i = 1,\dots,I,$$

where ϕ_j is a solution of (7) with $\phi_j(0)$ chosen small enough such that $\underline{U}_{j,i}(0) = \phi_j(a_i) \le \varphi_{j,i}$.
Verification of the inequalities:

- **Interior points ($2 \le i \le I-1$):** Here, $\Upsilon_{j,i} = 0$. We have:

  $$\underline{U}'_{j,i}(t) = \varepsilon\phi'_j(a_i + \varepsilon t).$$

  By Taylor expansion, $\delta^2\underline{U}_{j,i}(t) \approx \phi'_j(s)\cdot 2\kappa + \phi''_j(s)(2\kappa x_i)^2$ with $s = a_i + \varepsilon t$. Since $\phi''_j(s) \ge 0$ and $(2\kappa x_i)^2 \ge 0$, we have $\delta^2\underline{U}_{j,i}(t) \ge 2\kappa\phi'_j(s)$. Given that $\varepsilon \le 2\kappa$, it follows:

  $$\underline{U}'_{j,i}(t) \le 2\kappa\phi'_j(s) \le \delta^2\underline{U}_{j,i}(t).$$

  Thus, $\underline{U}'_{j,i}(t) \le \delta^2\underline{U}_{j,i}(t) + \Upsilon_{j,i}\underline{U}^{p_j}_{j,i}\underline{U}^{q_{j+1}}_{j+1,i}$.

- **Boundary points ($i = 1$ and $i = I$):** Here, $\Upsilon_{j,i} = 2/h$. We have:

$$\underline{U}'_{j,i}(t) = \varepsilon\phi'_j(s) \leq \frac{2}{h}\phi_j^{p_j}(s)\phi_{j+1}^{q_{j+1}}(s) = \frac{2}{h}\underline{U}_{j,i}^{p_j}\underline{U}_{j+1,i}^{q_{j+1}}.$$

Additionally, due to the strict growth of ϕ_j (from the ODE $\phi'_j > 0$) and the decrease of $a_i = \kappa x_i^2$ away from the boundaries (due to grid symmetry), $\underline{U}_{j,i}(t)$ is larger at the boundary points than at adjacent points, implying $\delta^2\underline{U}_{j,i}(t) \leq 0$ for $i = 1$ and $i = I$, so:

$$\underline{U}'_{j,i}(t) \leq \delta^2\underline{U}_{j,i}(t) + \frac{2}{h}\underline{U}_{j,i}^{p_j}\underline{U}_{j+1,i}^{q_{j+1}}.$$

Hence, $(\underline{U}_{1,h}, \ldots, \underline{U}_{n,h})$ is a lower solution of (2)–(4). Since $\phi_j(s)$ blows up in finite time and $s = a_i + \varepsilon t \geq \varepsilon t$, $\underline{U}_{j,i}(t)$ blows up in finite time. By the comparison principle (Lemma 2), $(U_{1,h}, \ldots, U_{n,h})$ also blows up in finite time.
This completes the proof.

Remark 1. *From inequality (6), we have:*

$$\frac{U'_{j,I}(t)}{U_{j,I}^{p_j}(t)} \geq \frac{1}{\beta_j}.$$

Integrating both sides over the interval $[t, T_h]$ for $t \in (0, T_h)$:

$$\int_t^{T_h} \frac{U'_{j,I}(\tau)}{U_{j,I}^{p_j}(\tau)} d\tau \geq \frac{1}{\beta_j}\int_t^{T_h} d\tau = \frac{T_h - t}{\beta_j}.$$

The left-hand side simplifies to:

$$\int_{U_{j,I}(t)}^{\infty} \frac{du}{u^{p_j}} = \frac{1}{(p_j - 1)U_{j,I}^{p_j-1}(t)}.$$

Thus,

$$\frac{1}{(p_j - 1)U_{j,I}^{p_j-1}(t)} \geq \frac{T_h - t}{\beta_j},$$

which implies:

$$\frac{1}{p_j - 1} \cdot \frac{1}{U_{j,I}^{p_j-1}(t)} \geq \gamma(T_h - t), \quad \text{where} \quad \gamma = \frac{1}{\beta_j}.$$

Consequently, there exists a constant $C_{p_j} = \left(\frac{\beta_j}{p_j - 1}\right)^{1/(p_j-1)}$ *such that:*

$$U_{j,I}(t) \le C_{p_j}(T_h - t)^{-1/(p_j-1)}, \quad t \in (0, T_h), \quad j = 1, 2, \dots, n,$$

provided that $p_j > 1$ *for all* j.
Theorems 1 and 2 yield the following corollaries:

Corollary 1. *The solution of* (2)–(4) *blows up in finite time* T_h *if*

$$\max\left\{p_j - 1\ (j = 1, 2, \dots, n),\ \prod_{j=1}^{n} q_j - \prod_{j=1}^{n}(1 - p_j)\right\} > 0.$$

Corollary 2. *If*

$$\max\left\{p_j - 1\ (j = 1, 2, \dots, n),\ \prod_{j=1}^{n} q_j - \prod_{j=1}^{n}(1 - p_j)\right\} \le 0,$$

then the solution of (2)–(4) *exists globally.*

5. Non-simultaneous and simultaneous blow-up of the semidiscrete solution

In this section, we consider the positive solution $(U_{1,h}, \dots, U_{n,h})$ of (2)–(4), with h fixed, and we provide sufficient conditions for the occurrence of non-simultaneous and simultaneous blow-up for all initial data. Theorems 3 and 4 below present results where $k + 1$ components (with $k \in \{0, 1, \dots, n-2\}$) blow up, while the remaining $(n - k - 1)$ components remain bounded.

Definition 3. *We say that the solution* $(U_{1,h}, \dots, U_{n,h})$ *of* (2)–(4) ***blows up simultaneously*** *in a finite time if there exists a finite time* $T_h > 0$ *such that for* $t \in [0, T_h)$, $\max\{\|U_{1,h}\|_\infty, \dots, \|U_{n,h}\|_\infty\} < \infty$ *and*

$$\limsup_{t \to T_h} \min\{\|U_{1,h}\|_\infty, \dots, \|U_{n,h}\|_\infty\} = +\infty$$

The time T_h *is called the* ***simultaneous blow-up time****.*

Definition 4. *We say that the solution* $(U_{1,h}, \dots, U_{n,h})$ *of* (2)–(4) ***blows up non-simultaneously*** *in finite time if there exists a time* $T_h > 0$ *and a subset* $W \subset \{1, \dots, n\}$ *such that:*

(1) *For all* $t \in [0, T_h)$, $\max\{\|U_{1,h}\|_\infty, \dots, \|U_{n,h}\|_\infty\} < \infty$,

(2) *For all* $j \in W$, $\limsup_{t \to T_h} \|U_{j,h}(t)\|_\infty = +\infty$,

(3) *For all* $j \notin W$, $\limsup_{t \to T_h} \|U_{j,h}(t)\|_\infty < \infty$.

The time T_h *is called the* ***non-simultaneous blow-up time****.*

Theorem 3. *Fix* $j \in \{1, \dots, n\}$ *and define* $\beta_j = \dfrac{1}{p_j - 1}$*. Assume* $p_m \leq 1 < p_j$ *for* $m = 1, 2, \dots, j-1, j+1, \dots, n$.

If $k \in \{0, 1, \dots, n-2\}$*, and for all* $\eta = j-1, j-2, \dots, j-k$*, the coefficients* $\beta_\eta = \dfrac{1 - q_{\eta+1}\beta_{\eta+1}}{p_\eta - 1}$ *satisfy* $\beta_\eta > 0$ *and* $p_\eta < 1$*, and if* $q_{j-k}\beta_{j-k} < 1$*, then the components* $U_{j-k,h}, U_{j-k+1,h}, \dots, U_{j,h}$ *blow up simultaneously in a finite time* T_h*, while the other* $(n-k-1)$ *components remain bounded. Moreover,*

$$(\|U_{j-k,h}(t)\|_\infty, \dots, \|U_{j,h}(t)\|_\infty) \sim \left((T_h - t)^{-\beta_{j-k}}, \dots, (T_h - t)^{-\beta_j}\right).$$

Without loss of generality, we prove the case $j = n$ using two lemmas. Thus, $\beta_n = \dfrac{1}{p_n - 1}$. The first lemma deals with the case $k = 0$.

Lemma 5. *Assume* $p_m \leq 1 < p_n$ *for* $m = 1, 2, \dots, n-1$*, and that* $q_n\beta_n < 1$*. Then only* $U_{n,h}$ *blows up in a finite time* T_h*, while the other components remain bounded. Moreover,*

$$\|U_{n,h}(t)\|_\infty \sim (T_h - t)^{-\beta_n}.$$

Proof. This proof consists of three steps.
Step 1. $U_{n,h}$ must be the blow-up component. Otherwise, $U_{1,h}, \dots, U_{n-1,h}$ would remain bounded also for $p_m \leq 1$ for all $m = 1, 2, \dots, n-1$, which contradicts the blow-up behavior of the full solution $U_{1,h}, \dots, U_{n,h}$, since $p_n > 1$. Consequently, $U_{n,h}$ blows up at T_h.
Step 2. $U_{1,h}, \dots, U_{n-1,h}$ remain bounded, and $\|U_{n,h}(t)\|_\infty \leq C_{p_n} (T_h - t)^{-\beta_n}$.
For $p_n > 1$, it follows from Remark 1 that

$$\|U_{n,h}(t)\|_\infty \leq C_{p_n} (T_h - t)^{-\beta_n}, \quad \forall t \in (0, T_h). \tag{8}$$

Suppose that $U_{n-1,h}$ blows up at T_h and that I is a blow-up node. From (2), we have

$$U'_{n-1,I}(t) = \frac{2U_{n-1,I-1}(t) - 2U_{n-1,I}(t)}{h^2} + \frac{2}{h} U^{p_{n-1}}_{n-1,I}(t) U^{q_n}_{n,I}(t), \quad t \in (0, T_h).$$

Since $U_{n-1,I-1}(t) < U_{n-1,I}(t)$, for all $t \in (0, T_h)$ (by Lemma 3), and using (8), we obtain

$$U'_{n-1,I}(t) \leq \frac{2}{h} U_{n-1,I}^{p_{n-1}}(t) (C_{p_n})^{q_n} (T_h - t)^{-q_n \beta_n}, \quad \forall t \in (t_0, T_h),$$

which implies that

$$\frac{U'_{n-1,I}(t)}{U_{n-1,I}^{p_{n-1}}(t)} \leq \frac{2}{h} (C_{p_n})^{q_n} (T_h - t)^{-q_n \beta_n}, \quad \forall t \in (t_0, T_h).$$

Integrating this inequality from t_0 to t, we obtain:
Case 1. $p_{n-1} = 1$

$$U_{n-1,I}(t) \leq \exp \left[\ln \left(U_{n-1,I}(t_0) \right) + C_1 T_h^{\dfrac{p_n - q_n - 1}{p_n - 1}} \right], \quad \forall t \in (t_0, T_h),$$

Case 2. $p_{n-1} < 1$

$$U_{n-1,I}(t) \leq \left[U_{n-1,I}^{1-p_{n-1}}(t_0) + (1 - p_{n-1}) C_1 T_h^{\dfrac{p_n - q_n - 1}{p_n - 1}} \right]^{\dfrac{1}{1 - p_{n-1}}}, \quad \forall t \in (t_0, T_h),$$

where $C_1 = \dfrac{2(C_{p_n})^{q_n} (p_n - 1)}{(p_n - q_n - 1)h}$.
which contradicts the assumption that $U_{n-1,h}$ blows up at T_h. Hence, by induction, $U_{j,h}$ remains bounded for all $j = n-2, n-3, \ldots, 1$, provided that $p_j \leq 1$.
Step 3. $\|U_{n,h}(t)\|_\infty \geq c_{p_n} (T_h - t)^{-\beta_n}$.
From (2), we have

$$U'_{n,I}(t) = \frac{2U_{n,I-1}(t) - 2U_{n,I}(t)}{h^2} + \frac{2}{h} U_{n,I}^{p_n}(t) U_{1,I}^{q_1}(t), \quad t \in (0, T_h).$$

As $U_{n,I-1}(t) < U_{n,I}(t), \quad \forall t \in (0, T_h)$ (Lemma 3), then

$$U'_{n,I}(t) \leq \frac{2}{h} U_{n,I}^{p_n}(t) U_{1,I}^{q_1}(t), \quad \forall t \in (0, T_h),$$

since $p_1 \leq 1$, there exists a constant $C > 0$ such that $U_{1,I}(t) \leq C$, $\forall t \in (0, T_h)$, then $U_{n,I}(t)$ satisfies

$$U'_{n,I}(t) \leq \frac{2}{h} C^{q_1} U_{n,I}^{p_n}(t), \quad \forall t \in (0, T_h),$$

which implies that

$$\frac{U'_{n,I}(t)}{U^{p_n}_{n,I}(t)} \leq \frac{2}{h}C^{q_1}, \quad \forall t \in (0, T_h),$$

integrating this inequality from t to T_h, we obtain

$$U_{n,I}(t) \geq c_{p_n} (T_h - t)^{-\dfrac{1}{p_n - 1}}, \quad \forall t \in [0, T_h),$$

where $c_{p_n} = \left[\frac{2}{h}C^{q_1}(p_n - 1)\right]^{-\dfrac{1}{p_n - 1}}$ and the proof is completed.
Next, we prove the case for $k = 1$. The remaining subcases $k \in \{2, 3, \ldots, n-2\}$ can be treated similarly.

Lemma 6. *If $p_m \leq 1 < p_n$ for $m = 1, 2, \ldots, n-2$, $p_{n-1} < 1$, $\beta_{n-1} = \dfrac{1 - q_n\beta_n}{p_{n-1} - 1} > 0$, and $1 - q_{n-1}\beta_{n-1} > 0$, then $U_{n-1,h}$ and $U_{n,h}$ blow up simultaneously in finite time T_h, while the other $(n-2)$ components remain bounded. Moreover,*

$$(\|U_{n-1,h}(t)\|_\infty, \|U_{n,h}(t)\|_\infty) \sim \left((T_h - t)^{-\beta_{n-1}}, (T_h - t)^{-\beta_n}\right).$$

Proof. This proof is divided into four steps.
Step 1. Both $U_{n-1,h}$ and $U_{n,h}$ are the blow-up components. We claim that $U_{n,h}$ is the blow-up component. If not, then all other components would remain bounded, since $p_m \leq 1$ for $m = 1, 2, \ldots, n-2$ and $p_{n-1} < 1$, which contradicts the fact that $p_n > 1$. Consequently, $U_{n,h}$ blows up at time T_h.
Assume that $U_{n-1,h}$ remains bounded up to time T_h. Then, $U_{1,h}, \ldots, U_{n-2,h}$ would also remain bounded. Since $U_{n,h}$ blows up in T_h, by Step 3 of Lemma 5, we have

$$U_{n,I}(t) \geq c_{p_n} (T_h - t)^{-\beta_n}, \quad \forall t \in [0, T_h). \tag{9}$$

From (2) and (9), we have

$$U'_{n-1,I}(t) \geq \frac{2U_{n-1,I-1}(t) - 2U_{n-1,I}(t)}{h^2} + \frac{2}{h}U^{p_{n-1}}_{n-1,I}(t)(c_{p_n})^{q_n} (T_h - t)^{-\beta_n q_n}, \quad t \in (0, T_h).$$

By a method similar to the proof of Theorem 1, it is easy to see that

$$U'_{n-1,I}(t) \geq \frac{2}{h}U^{p_{n-1}}_{n-1,I}(t)(c_{p_n})^{q_n} (T_h - t)^{-\beta_n q_n}, \quad \forall t \in (t_0, T_h),$$

since $U_{n-1,I}$ is bounded, there exists a constant $C > 0$ such that

$$U'_{n-1,I}(t) \;\; \geq \;\; \frac{2}{h} C^{p_{n-1}} (c_{p_n})^{q_n} \left(T_h - t\right)^{-\beta_n q_n}, \quad \forall t \in (t_0, T_h),$$

thus

$$U'_{n-1,I}(t) \;\; \geq \;\; C_1 \left(T_h - t\right)^{-\beta_n q_n}, \quad \forall t \in (t_0, T_h),$$

with $C_1 = \dfrac{2B}{h} C^{p_{n-1}} (c_{p_n})^{q_n}$.
Integrating this inequality from t_0 to T_h, we have

$$U_{n-1,I}(T_h) \;\; \geq \;\; U_{n-1,I}(t_0) + C_1 \int_{t_0}^{T_h} \left(T_h - t\right)^{-\beta_n q_n} dt.$$

The boundedness of $U_{n-1,h}$ requires that $q_n \beta_n < 1$, which implies $\beta_{n-1} < 0$, contradicting the assumption that $\beta_{n-1} > 0$.
Step 2. Upper blow-up rate estimates of $U_{n-1,h}$ and $U_{n,h}$.
For $p_n > 1$, we have from Remark 1,

$$\|U_{n,h}(t)\|_\infty \leq C_{p_n} \left(T_h - t\right)^{-\beta_n}, \quad \forall t \in (0, T_h). \tag{10}$$

From (2), we have

$$U'_{n-1,I}(t) = \frac{2U_{n-1,I-1}(t) - 2U_{n-1,I}(t)}{h^2} + \frac{2}{h} U^{p_{n-1}}_{n-1,I}(t) U^{q_n}_{n,I}(t), \quad t \in (0, T_h).$$

Since $U_{n-1,I-1}(t) < U_{n-1,I}(t)$, for all $t \in (0, T_h)$ (Lemma 3), and using (10), we obtain

$$U'_{n-1,I}(t) \leq \frac{2}{h} U^{p_{n-1}}_{n-1,I}(t) (C_{p_n})^{q_n} \left(T_h - t\right)^{-q_n \beta_n}, \quad \forall t \in (t_0, T_h),$$

which implies that

$$\frac{U'_{n-1,I}(t)}{U^{p_{n-1}}_{n-1,I}(t)} \;\; \leq \;\; \frac{2}{h} (C_{p_n})^{q_n} \left(T_h - t\right)^{-q_n \beta_n}, \quad \forall t \in (t_0, T_h).$$

Integrating this inequality from t_0 to t, we obtain

$$U^{1-p_{n-1}}_{n-1,I}(t) \leq U^{1-p_{n-1}}_{n-1,I}(t_0) + \frac{2\left(1 - p_{n-1}\right)\left(C_{p_n}\right)^{q_n}}{h\left(1 - q_n \beta_n\right)} \left(T_h - t_0\right)^{1 - q_n \beta_n} +$$
$$\frac{2\left(p_{n-1} - 1\right)\left(C_{p_n}\right)^{q_n}}{h\left(1 - q_n \beta_n\right)} \left(T_h - t\right)^{1 - q_n \beta_n}, \quad \forall t \in (t_0, T_h),$$

since $p_{n-1} < 1$.
As $\dfrac{1-q_n\beta_n}{p_{n-1}-1} > 0$ and $p_{n-1}-1<0$, it follows that $1-q_n\beta_n<0$. Thus $\dfrac{2\left(1-p_{n-1}\right)\left(C_{p_n}\right)^{q_n}}{h\left(1-q_n\beta_n\right)} < 0$.
Hence, there exists a constant $C>0$ such that

$$U_{n-1,I}^{1-p_{n-1}}(t) \leq \frac{2C\left(p_{n-1}-1\right)\left(C_{p_n}\right)^{q_n}}{h\left(1-q_n\beta_n\right)}\left(T_h-t\right)^{1-q_n\beta_n}, \quad \forall t\in(0,T_h),$$

which implies that

$$U_{n-1,I}(t) \leq C_{p_{n-1}}\left(T_h-t\right)^{-\beta_{n-1}}, \quad \forall t\in(0,T_h),$$

where $C_{p_{n-1}} = \left[\dfrac{2C\left(p_{n-1}-1\right)\left(C_{p_n}\right)^{q_n}}{h\left(1-q_n\beta_n\right)}\right]^{\dfrac{1}{1-p_{n-1}}}$.

Step 3. Boundedness of $U_{1,h},\ldots,U_{n-2,h}$. This part is similar to Step 2 of Lemma 5.

Step 4. Lower blow-up rate estimates for $U_{n-1,h}$ and $U_{n,h}$.
Since $U_{n,h}$ blows up at T_h, by Step 3 of Lemma 5 we have

$$U_{n,I}(t) \geq c_{p_n}\left(T_h-t\right)^{-\beta_n}, \quad \forall t\in[0,T_h). \tag{11}$$

From (2) and (11), we have

$$U'_{n-1,I}(t) \geq \frac{2U_{n-1,I-1}(t)-2U_{n-1,I}(t)}{h^2}+\frac{2}{h}U_{n-1,I}^{p_{n-1}}(t)(c_{p_n})^{q_n}\left(T_h-t\right)^{-\beta_nq_n}, \quad t\in(0,T_h).$$

By a method similar to the proof of Theorem 1, it is easy to see that

$$U'_{n-1,I}(t) \;\; \geq \;\; \frac{2}{h}U_{n-1,I}^{p_{n-1}}(t)(c_{p_n})^{q_n}\left(T_h-t\right)^{-\beta_nq_n}, \quad \forall t\in(z,T_h).$$

Integrating the above inequality from z to t, we obtain

$$U_{n-1,I}(t) \;\; \geq \;\; \frac{2}{h}(c_{p_n})^{q_n}\int_z^t U_{n-1,I}^{p_{n-1}}(\tau)\left(T_h-\tau\right)^{-\beta_nq_n}d\tau, \quad \forall t\in(z,T_h).$$

Define $H(t)=\int_z^t U_{n-1,I}^{p_{n-1}}(\tau)\left(T_h-\tau\right)^{-\beta_nq_n}d\tau$. Then $U_{n-1,I}^{-p_{n-1}}(t)H'(t)=\left(T_h-t\right)^{-\beta_nq_n}$.
Since

$$U_{n-1,I}(t) \;\; \geq \;\; \frac{2}{h}(c_{p_n})^{q_n}H(t), \quad \forall t\in(z,T_h), \tag{12}$$

then

$$H^{-p_{n-1}}(t)H'(t) \;\; \geq \;\; Q\,(T_h - t)^{-\beta_n q_n}\,, \quad \forall t \in (z, T_h),$$

where $Q = \left(\dfrac{2}{h}(c_{p_n})^{q_n}\right)^{p_{n-1}}$.

Integrating the above inequality from z to t and taking $z = 2t - T_h$, we have

$$H^{1-p_{n-1}}(t) \;\; \geq \;\; (1 - p_{n-1})\,K\,(T_h - t)^{1-q_n\beta_n}\,, \quad \forall t \in (0, T_h), \tag{13}$$

where $K = \dfrac{2^{1-q_n\beta_n}Q}{(1 - q_n\beta_n)} - \dfrac{Q}{(1 - q_n\beta_n)}$. From (12), we deduce that

$$U_{n-1,I}^{1-p_{n-1}}(t) \;\; \geq \;\; \left(\frac{2}{h}(c_{p_n})^{q_n}\right)^{1-p_{n-1}} H^{1-p_{n-1}}(t), \quad \forall t \in (0, T_h). \tag{14}$$

Using (13) and (14), we obtain

$$U_{n-1,I}^{1-p_{n-1}}(t) \;\; \geq \;\; K\left(\frac{2}{h}(c_{p_n})^{q_n}\right)^{1-p_{n-1}}(1 - p_{n-1})\,(T_h - t)^{1-q_n\beta_n}\,, \quad \forall t \in (0, T_h),$$

and hence

$$U_{n-1,I}(t) \;\; \geq \;\; c_{p_{n-1}}\,(T_h - t)^{-\beta_{n-1}}\,, \quad \forall t \in (0, T_h),$$

where $c_{p_{n-1}} = \left[K\left(\dfrac{2}{h}(c_{p_n})^{q_n}\right)^{1-p_{n-1}}(1 - p_{n-1})\right]^{\dfrac{1}{1-p_{n-1}}}$, and the proof is completed.

Theorem 4. *Fix $j \in \{1, \dots, n\}$ and define $\beta_j = \dfrac{1}{p_j - 1}$. Assume that $p_m \leq 1 < p_j$ for all $m = 1, 2, \dots, j-1, j+1, \dots, n$. If $k = n-1$, $\beta_\eta = \dfrac{1 - q_{\eta+1}\beta_{\eta+1}}{p_\eta - 1}$, and $p_\eta < 1$ for all $\eta = j-1, j-2, \dots, j+1-n$, and $\beta_\eta > 0$ for all $\eta = j-1, j-2, \dots, j+2-n$, and $\beta_{j+1-n} \geq 0$, then $U_{1,h}, \dots, U_{n,h}$ blow up simultaneously in finite time T_h.*

Without loss of generality, we prove the case $j = n$. So, $\beta_n = \dfrac{1}{p_n - 1}$.

Lemma 7. *If $\beta_\eta = \dfrac{1 - q_{\eta+1}\beta_{\eta+1}}{p_\eta - 1}$ and $p_\eta \leq 1 < p_n$ for all $\eta = 1, 2, \dots, n-1$, and if $\beta_1 \geq 0$ and $\beta_\eta > 0$ for all $\eta = 2, 3, \dots, n-1$, then $U_{1,h}, \dots, U_{n,h}$ blow up* *simultaneously in finite time* T_h.

Proof. Similarly to Step 1 of Lemma 6, we show that $U_{n-1,h}$ and $U_{n,h}$ blow up simultaneously in finite time T_h. By Step 4 of Lemma 6, we have $U_{n-1,I}(t) \geq c_{p_{n-1}}(T_h - t)^{-\beta_{n-1}}$. Using a similar method, we find that $U_{m,h}$ blows up in finite time T_h, and that $U_{m,I}(t) \geq c_{p_m}(T_h - t)^{-\beta_m}$ for all $m = n-2, n-3, \ldots, 2$. Similarly, for $\beta_1 \geq 0$, $U_{1,h}$ also blows up at time T_h. This shows that $U_{1,h}, \ldots, U_{n,h}$ blow up simultaneously in finite time T_h, and the proof is complete.

6. Convergence of semidiscrete blow-up time

In this section, we study the convergence of the semidiscrete blow-up time. We now show that for each fixed time interval $[0, T^*]$ on which the solution $(u_1, \ldots, u_n)$ is defined, the semidiscrete solution $(U_{1,h}, \ldots, U_{n,h})$ approximates $(u_1, \ldots, u_n)$ when the mesh parameter h tends to zero. We denote

$$u_{j,h}(t) \quad = \quad (u_j(x_1, t), \ldots, u_j(x_I, t))^T, \quad j = 1, \ldots, n.$$

Theorem 5. *Assume that the problem* (1) *has a solution* $(u_j)_{j=1}^n$*, where* $u_j \in C^{4,1}([-1,1] \times [0, T^*])$ *for* $j = 1, \ldots, n$*, and that the initial data* $\varphi_{j,h}$ *of* (2)–(4) *satisfy*

$$\|\varphi_{j,h} - u_j(0)\|_\infty = o(1), \quad h \to 0, \quad j = 1, \ldots, n. \tag{15}$$

Then, for h sufficiently small, the problem (2)–(4) *has a unique solution* $(U_{j,h})_{j=1}^n$*, where* $U_{j,h} \in C^1\left([0, T^*], \mathbb{R}^I\right)$ *for* $j = 1, \ldots, n$*, such that*

$$\max_{t \in [0,T^*]} \|U_{j,h}(t) - u_{j,h}(t)\|_\infty = O\left(\sum_{b=1}^{n} \|\varphi_{b,h} - u_{b,h}(0)\|_\infty + h\right), \ h \to 0, \ j = 1, \ldots, n.$$

Proof. Let $\nu > 0$ be such that

$$\max\left(\|u_1(\cdot, t)\|_\infty, \ldots, \|u_n(\cdot, t)\|_\infty\right) < \nu, \quad t \in [0, T^*]. \tag{16}$$

Let $t(h) \leqslant T^*$ be the greatest value of $t > 0$ such that

$$\max_{1 \leqslant j \leqslant n} \left\{\|U_{j,h}(t) - u_{j,h}(t)\|_\infty\right\} < 1, \quad t \in (0, t(h)). \tag{17}$$

The relation (15) implies $t(h) > 0$, for h small enough. Using the triangle inequality, we obtain

$$\|U_{j,h}(t)\|_\infty \leqslant 1 + \nu, \quad j = 1, \ldots, n, \quad \text{for } t \in (0, t(h)). \tag{18}$$

Let $e_{j,i}(t) = U_{j,i}(t) - u_{j,i}(t)$ denote the discretization error for $j = 1, \ldots, n$, $i = 1, \ldots, I$ and $t \in [0, T^*]$.
Let $X_1 \in C^{4,1}([-1,1] \times [0, T^*])$ be such that

$$X_1(x,t) = \left(\sum_{b=1}^{n} \|\varphi_{b,h} - u_{b,h}(0)\|_\infty + Kh \right) \mathrm{e}^{(M+3)t + x^2 - 1},$$

and define $X_1 = \ldots = X_n$ for all $(x,t) \in [-1,1] \times [0, T^*]$, where K and M are positive constants.

By the Lemma 2, we can prove that

$$|e_{j,i}(t)| < X_j(x_i, t) \leq \max X_j(x_i, t), \text{for all } \ j = 1, \ldots, n, \ i = 1, \ldots, I \text{ and } t \in (0, t(h)).$$

Thus, we get

$$\|U_{j,h}(t) - u_{j,h}(t)\|_\infty \leqslant \left(\sum_{b=1}^{n} \|\varphi_{b,h} - u_{b,h}(0)\|_\infty + Kh \right) \mathrm{e}^{(M+3)t}, \ j = 1, \ldots, n \ t \in (0, t(h)).$$

Suppose that $T^* > t(h)$. From (17), we obtain

$$1 = \|U_{j,h}(t(h)) - u_{j,h}(t(h))\|_\infty \leq \left(\sum_{b=1}^{n} \|\varphi_{b,h} - u_{b,h}(0)\|_\infty + Kh \right) \mathrm{e}^{(M+3)T^*} \ j = 1, \ldots, n.$$

Since the right-hand side of the inequality tends to zero as $h \to 0$, we obtain the contradiction $1 \leqslant 0$, which is impossible. Consequently, $t(h) = T^*$ and the proof is complete.

Theorem 6. *Suppose that the solution $(u_1, \ldots, u_n)$ of problem (1) blows up in a finite time T such that $u_j \in C^{4,1}([-1,1] \times [0,T))$, $j = 1, \ldots, n$ and the initial data at (2)–(4) satisfies (15).*
Under the assumptions of Corollary 1, the solution $(U_{1,h}, \ldots, U_{n,h})$ of problem (2)–(4) blows up in a finite time T_h and we have

$$\lim_{h \to 0} T_h = T.$$

Proof. We consider the case $p_1 > 1$. The other cases $p_j > 1$ for $j \in \{2, \ldots, n\}$ can be handled similarly.

Let $\mu > 0$, there exists a constant $\kappa > 0$ such that

$$\frac{y^{1-p_1}}{\gamma\,(p_1 - 1)} \leqslant \frac{\mu}{2}, \quad \kappa \leqslant y. \tag{19}$$

Since u_1 blows up in a finite time T, there exists a time $T_0 \in (T - \mu/2; T)$ such that
$\|u_1(.,t)\|_\infty \geq 2\kappa$, for $t \in [T_0, T)$. Denote $T_1 = \dfrac{T_0 + T}{2}$, we see easily that $\sup_{t\in[0,T_1]} \|u_1(.,t)\|_\infty < \infty$. It follows from Theorem 5 that, for h sufficiently small,

$$\sup_{t\in[0,T_1]} \|U_{1,h}(t) - u_{1,h}(t)\|_\infty \leqslant \kappa.$$

Applying the triangle inequality, we get

$$\|U_{1,h}(T_1)\|_\infty \geq \|u_{1,h}(T_1)\|_\infty - \|U_{1,h}(T_1) - u_{1,h}(T_1)\|_\infty \geq \kappa.$$

From Corollary 1, we know that $(U_{1,h}, \dots, U_{n,h})$ blows up at the time T_h . We deduce from Remark 1 and from (19) that

$$|T_h - T| \leqslant |T_h - T_1| + |T_1 - T| \leqslant \frac{\|U_{1,h}(T_1)\|_\infty^{1-p_1}}{\gamma(p_1 - 1)} + \frac{\mu}{2} \leqslant \mu.$$

7. Numerical experiments

In this section, we present some numerical approximations of the blow-up time for system (1) and discuss the numerical results. We use the initial data

$$u_{j,0}(x) = \frac{1}{2} + \frac{1}{2}x^2 \quad j = 1, \dots, n.$$

Let η be the arc length along the curve $P(t) = (t, F_h(t))$, for all $t \in [0, T_h)$, where

$$F_h = (U_{1,h}, \dots, U_{n,h})^T, \quad F_h(0) = (\varphi_{1,h}, \dots, \varphi_{n,h})^T.$$

We consider t and F_h as functions of η. Since the arc length satisfies

$$d\eta^2 = dt^2 + dF_1^2 + \cdots + dF_{nI}^2,$$

the functions $t(\eta)$ and $F_h(\eta)$ satisfy the following system of differential equations:

$$\begin{cases} \dfrac{dt}{d\eta} = \dfrac{1}{\sqrt{1 + \sum\limits_{i=1}^{nI} f_i^2}}, \\ \dfrac{dF_i}{d\eta} = \dfrac{f_i}{\sqrt{1 + \sum\limits_{i=1}^{nI} f_i^2}}, \quad i = 1, \dots, nI, \\ t(0) = 0, \quad F_i(0) \geq 0, \quad i = 1, \dots, nI, \end{cases} \tag{20}$$

where $0 < \eta < \infty$ and

$$f_{jI+1} = \frac{2}{h^2}(U_{jI+2} - U_{jI+1}) + \frac{2}{h}U_{jI+1}^{p_{j+1}}U_{(j+1)I+1}^{q_{j+2}}, \quad j \in \{0, 1, \ldots, n-1\},$$

$$f_i = \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2}, \quad i \in \bigcup_{k=0}^{n-1}[kI+2, (k+1)I-1],$$

$$f_{(j+1)I} = \frac{2}{h^2}(U_{(j+1)I-1} - U_{(j+1)I}) + \frac{2}{h}U_{(j+1)I}^{p_{j+1}}U_{(j+2)I}^{q_{j+2}}, \quad j \in \{0, 1, \ldots, n-1\},$$

$$U_{(n+1)I} = U_I, \quad U_{(n+1)I+1} = U_1, \quad q_{n+1} = q_1,$$

with f_{jI+1} and $f_{(j+1)I}$ being the f_i at the boundaries.

It is well known (Hirota & Ozawa, 2006) that

$$\lim_{\eta\to\infty} t(\eta) = T_h, \quad \text{and} \quad \lim_{\eta\to\infty} \|F_h(\eta)\|_\infty = \infty.$$

For the numerical computations, we define a discrete sequence of arc lengths $\eta_l = 2^5 \cdot 2^l$, for $l = 0, 1, 2, \ldots, 10$. For each l, we apply the DOP54 method (see Hairer, Nørsett and Wanner, 1993) to system (20) up to $\eta = \eta_l$ and record the corresponding time approximation $t_l^{(0)} = t_l$. The resulting sequence $\{t_l^{(0)}\}_{l=0}^{10}$ converges linearly toward the blow-up time T_h. We further accelerate this convergence recursively using Aitken's Δ^2 method, which generates an improved sequence $\{t_l^{(k+1)}\}$ according to the rule:

$$t_{l+2}^{(k+1)} = t_{l+1}^{(k)} - \frac{(t_{l+2}^{(k)} - t_{l+1}^{(k)})^2}{t_{l+2}^{(k)} - 2t_{l+1}^{(k)} + t_l^{(k)}}, \quad l \geq 2k, \ k = 0, 1, 2, \ldots$$

As in Hirota & Ozawa (2006), we set `RTOL` = `ATOL` = 10^{-15} and `ITOL` = 10^{-8} for all experiments, where `RTOL` and `ATOL` denote the relative and absolute error tolerances, respectively, and `ITOL` determines how the errors are controlled.

7.1. Numerical results

The discrete initial conditions are given by

$$\varphi_{j,i} = \frac{1}{2} + \frac{1}{2}(ih - h - 1)^2, \quad j = 1, \ldots, n, \quad i = 1, \ldots, I.$$

In the following tables, we report the numerical blow-up times T_h, the number of iterations v, the absolute errors $E_h = |T_h - T_{2h}|$, and the observed orders of convergence s, for different spatial discretizations $I = 16, 32, 64, 128, 256, 512$.

The convergence order s is estimated by

$$s = \frac{\log\left((T_{4h} - T_{2h})/(T_{2h} - T_h)\right)}{\log(2)}.$$

Table 1: Lemma 5 with $n = 2$, $p_1 = 0.5$, $p_2 = 2.5$, $q_1 = 1$, $q_2 = 0.1$

I	T_h	v	E_h	s
16	0.169218174193191	2027	–	–
32	0.156559254478750	3158	1.266e-2	–
64	0.152408730825611	5386	4.151e-3	1.61
128	0.151116417563327	9723	1.292e-3	1.68
256	0.150728460698751	18210	3.880e-4	1.72
512	0.150613324963546	35259	1.151e-4	1.75

Table 2: Lemma 6 with $n = 3$, $p_1 = 0.5$, $p_2 = 0.5$, $p_3 = 2.5$, $q_1 = 2$, $q_2 = 0.2$, $q_3 = 2$

I	T_h	v	E_h	s
16	0.142024555904137	1887	–	–
32	0.130509786030290	2900	1.151e-2	–
64	0.126655288674220	4902	3.855e-3	1.58
128	0.125438120562574	8801	1.217e-3	1.66
256	0.125069058468841	16418	3.691e-4	1.72
512	0.124958331067865	31652	1.107e-4	1.74

Table 3: Lemma 7 with $n = 2$, $p_1 = 0.5$, $p_2 = 2.5$, $q_1 = 1$, $q_2 = 3$

I	T_h	v	E_h	s
16	0.115742236010809	1544	–	–
32	0.104475544720172	2396	1.127e-2	–
64	0.100734875763380	4155	3.741e-3	1.59
128	0.099564333761427	7325	1.171e-3	1.68
256	0.099211550354244	13701	3.528e-4	1.73
512	0.099108047323709	26118	1.035e-4	1.77

In addition to the numerical results presented in Tables 1 to 3, we provide figures illustrating the evolution of the different components, which help to better visualize the phenomena of simultaneous and non-simultaneous blow-up.

For the different plots, we used the same values of p_j and q_j $(j = 1, \ldots, n)$ used to obtain the tables above in the case where $I = 16$.

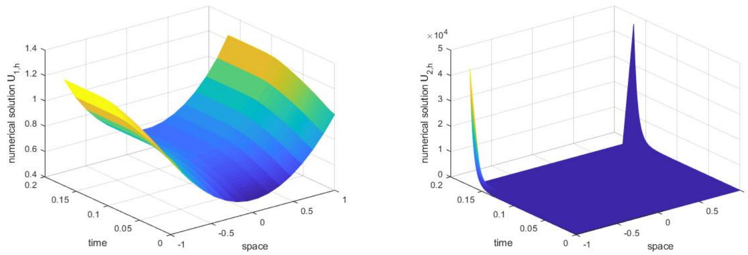

Figure 1: Lemma 5 : $U_{2,h}$ blows up while $U_{1,h}$ remains bounded for $n = 2$, $p_1 = 0.5$, $p_2 = 2.5$, $q_1 = 1$, $q_2 = 0.1$

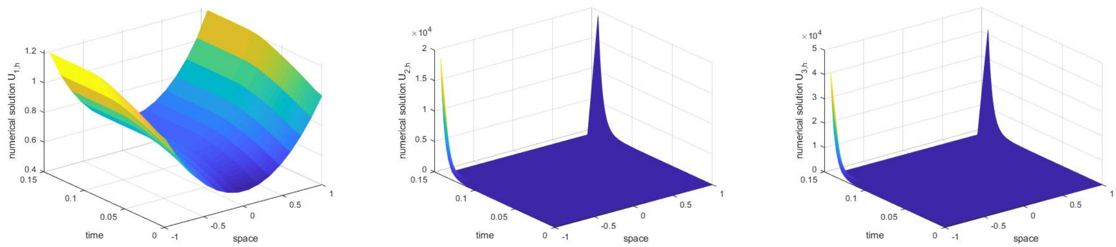

Figure 2: Lemma 6: $U_{3,h}$ and $U_{2,h}$ blow up while $U_{1,h}$ remains bounded for $n = 3$, $p_1 = 0.5$, $p_2 = 0.5$, $p_3 = 2.5$, $q_1 = 2$, $q_2 = 0.2$, $q_3 = 2$

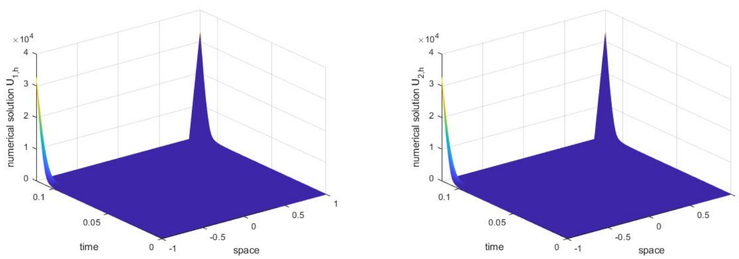

Figure 3: Lemma 7: $U_{1,h}$ and $U_{2,h}$ blow up simultaneously for $n = 2$, $p_1 = 0.5$, $p_2 = 2.5$, $q_1 = 1$, $q_2 = 3$

7.2. Discussion of numerical results

The numerical experiments robustly confirm the theoretical findings. The estimated blow-up times T_h converge with a consistent order $s \approx 1.7$, supporting the theoretical convergence analysis (Section 6). The rapid decrease of the absolute error E_h with mesh refinement provides clear quantitative evidence of this convergence. The application of Aitken's Δ^2 method was instrumental in this process, as it efficiently accelerated the initially linear convergence of the iterative sequence, enabling a precise and computationally affordable estimation of T_h without compromising accuracy.

Figures 1 to 3 perfectly illustrate the theoretical regimes: non-simultaneous blow-up (Figs. 1, 2) and simultaneous blow-up (Fig. 3), in exact agreement with the conditions of Lemmas 5 to 7. Furthermore, all simulations validate the theoretical characterization of the blow-up set for the continuous problem (Theorem 4.4, [16]), which states that blow-up can only occur on the boundary. In our numerical solutions, the explosive growth is observed exclusively at the boundary nodes ($x = \pm 1$), while the solution remains bounded in the interior. This confirms the scheme's ability to correctly capture the spatial localization of the singularity.

8. Conclusion

This paper presents a comprehensive numerical study of blow-up phenomena for a system of n heat equations with nonlinear boundary conditions. In particular, it provides the first complete numerical analysis of non-simultaneous blow-up for such a multi-component system. A semi-discrete finite difference scheme is proposed and shown to preserve the essential blow-up properties of the continuous problem. Theoretical conditions characterizing simultaneous and non-simultaneous blow-up are established, and the convergence of the numerical blow-up time is proved. Numerical experiments fully support the theoretical analysis and confirm that blow-up occurs only at the boundary, in agreement with Theorem 4.4 [16]. The numerical strategy based on arc-length reparametrization, combined with the DOP54 solver and Aitken's acceleration, proves to be both robust and efficient.

Future work may include the study of fully discrete schemes, extensions to more general nonlinearities or higher-dimensional domains, and a detailed numerical analysis of blow-up profiles.

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