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FUZZY METRIC APPROACH TO (κ, α, β)-INTERPOLATIVE KANNAN CONTRACTIONS AND NONLINEAR INTEGRAL EQUATIONS

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Abstract: In this paper, we extend the concept of interpolative Kannan-type contractions to complete fuzzy metric spaces and establish the existence and uniqueness of fixed points. A Picard iteration scheme is shown to converge to the unique fixed point. An illustrative example involving a nonlinear integral equation demonstrates the applicability of the main result.

Keywords and Phrases: Fuzzy metric space, interpolative Kannan contraction, fixed point.

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1. Introduction

Fixed point theory plays a fundamental role in nonlinear analysis and has wide-ranging applications in areas such as optimization, control theory, differential equations, and computer science. Since the introduction of Banach's Contraction Principle [2], many generalizations have been proposed in metric fixed point theory.

Among these, Kannan's contraction [9, 10] is particularly significant because it guarantees the existence of fixed points even for certain discontinuous mappings.

A notable recent development in this direction was introduced by Karapınar [6] in the form of an interpolative Kannan-type contraction. It was shown in [6] (see also [1, 4, 7, 8]) that every such mapping admits at least one fixed point in a complete metric space. More precisely, we have the following result.

Theorem 1.1. (Karapınar [6]) *Let* (X, d) *be a complete metric space and let* $T : X \to X$ *be a self mapping satisfying the interpolative Kannan-type contractive condition, that is, there exist constants* $\kappa \in [0, 1)$ *and* $\gamma \in (0, 1)$ *such that*

$$d(Tx, Ty) \leq \kappa\, d(x, Tx)^{\gamma}\, d(y, Ty)^{1-\gamma}$$

for all $x, y \in X \setminus \mathrm{Fix}(T)$*, where*

$$\mathrm{Fix}(T) = \{x \in X : Tx = x\}.$$

Then the mapping T *possesses at least one fixed point in* X*.*

In parallel with these advances, the theory of fuzzy metric spaces, initiated by Kramosil and Michalek [11] and later refined by George and Veeramani [5], has provided a powerful framework for handling uncertainty and vagueness in metric concepts. This framework has proven to be particularly useful in applications involving imprecise data, such as decision-making, pattern recognition, and artificial intelligence.

2. Preliminaries

Now, we begin with some basic definitions.

Definition 2.1. *A binary operation* $*\colon [0, 1] \times [0, 1] \to [0, 1]$ *is called a continuous* t*-norm if it satisfies the following conditions:*

(i) $*$ *is commutative and associative;*

(ii) $*$ *is continuous;*

(iii) $a * 1 = a, \ \forall a \in [0, 1]$;

(iv) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$, *for all* $a, b, c, d \in [0, 1]$.

Example 2.2. $a * b = \min\{a, b\}$ and $a * b = ab$ are t-norms.

The concept of fuzzy metric spaces was introduced by Kramosil and Michalek [11] as follows:

Definition 2.3. *A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$ and $s, t > 0$:*

(i) $M(x, y, 0) = 0$;

(ii) $M(x, y, t) = 1$ *for all* $t > 0$ *if and only if* $x = y$;

(iii) $M(x, y, t) = M(y, x, t)$ *(symmetry);*

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(v) $M(x, y, \cdot)\colon (0, \infty) \to (0, 1]$ *is continuous.*

Note that $M(x, y, t)$ can be interpreted as the degree of nearness between x and y with respect to t.

Example 2.4. Let (X, d) be a metric space. Define $a * b = ab$ for all $a, b \in [0, 1]$. Define

$$M(x, y, t) = \frac{t}{t + d(x, y)}, \quad \forall x, y \in X, \ t > 0.$$

Then $(X, M, *)$ is a fuzzy metric space. This fuzzy metric induced by the metric d is called the standard fuzzy metric.

Definition 2.5. *Let $(X, M, *)$ be a fuzzy metric space. Then*

(a) a sequence $\{x_n\}$ in X is said to

(i) be a Cauchy sequence if

$$\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1, \quad \forall t > 0, \ n, p \in \mathbb{N};$$

(ii) be convergent to a point $x \in X$ if

$$\lim_{n \to \infty} M(x_n, x, t) = 1, \quad \forall t > 0.$$

(b) X is said to be complete if every Cauchy sequence in X converges to some point in X.

Example 2.6. Let $X = [0, 1]$ and let $*$ be the continuous t-norm defined by $a * b = ab$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$ and $x, y \in X$, define

$$M(x, y, t) = \begin{cases} \dfrac{t}{t + |x - y|^2}, & \text{if } t > 0, \\ 0, & \text{if } t = 0. \end{cases}$$

Clearly, $(X, M, *)$ is a complete fuzzy metric space.

Inspired by Karapınar et al. [6], Prachi et al. [12] introduced the concept of fuzzy interpolative Kannan-type contractions.

Definition 2.7. *Let $(X, M, *)$ be a complete fuzzy metric space. A mapping $T\colon X \to X$ is said to be a fuzzy interpolative Kannan-type contraction on X if there exist constants $\kappa \in [0, 1)$, $\gamma \in (0, 1)$, and $t > 0$ such that*

$$M(Tx, Ty, t) > \kappa \left[M(x, Tx, t)^{\gamma} * M(y, Ty, t)^{1-\gamma} \right],$$

for all $x, y \in X \setminus \mathrm{Fix}(T)$, where

$$\mathrm{Fix}(T) = \{x \in X : Tx = x\}.$$

Theorem 2.8. *Let $(X, M, *)$ be a complete fuzzy metric space. If $T\colon X \to X$ is a fuzzy interpolative Kannan-type contraction, then T has a unique fixed point in X.*

3. Main Results

We start with the following definition.

Definition 3.1. *Let $(X, M, *)$ be a fuzzy metric space and let $T : X \to X$ be a self-mapping. Then T is called a (κ, α, β)-interpolative Kannan-type contraction if there exist constants $\kappa \in (0, 1)$ and $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$ such that*

$$M(Tx, Ty, t) \geq \left[M(x, Tx, t)^{\alpha} * M(y, Ty, t)^{\beta} \right]^{\kappa}$$

for all $x, y \in X$ with $x \neq Tx$ and $y \neq Ty$, and for all $t > 0$, where $*$ is a continuous t-norm.

Theorem 3.2. *Let $(X, M, *)$ be a complete fuzzy metric space and assume that $*$ is the product t-norm, that is, $a * b = a \cdot b$. Let $T : X \to X$ be a (κ, α, β)-interpolative Kannan-type contraction. Then there exist constants $\kappa \in (0, 1)$ and $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$ such that for all $x, y \in X$ with $x \neq Tx$ and $y \neq Ty$ and for every $t > 0$,*

$$M(Tx, Ty, t) \geq \Big(M(x, Tx, t)^{\alpha}\, M(y, Ty, t)^{\beta} \Big)^{\kappa}.$$

Then T has a unique fixed point $x^* \in X$.

Proof. Fix an arbitrary $x_0 \in X$ and define the Picard sequence

$$x_{n+1} = Tx_n, \qquad n = 0, 1, 2, \ldots.$$

For simplicity, for each fixed $t > 0$, set

$$a_n := M(x_n, x_{n+1}, t) = M(Tx_{n-1}, Tx_n, t), \qquad n \geq 1.$$

By the contractive hypothesis we have, for every $n \geq 1$,

$$a_{n+1} = M(Tx_n, Tx_{n+1}, t) \geq \big(M(x_n, x_{n+1}, t)^{\alpha}\, M(x_{n+1}, x_{n+2}, t)^{\beta}\big)^{\kappa} = \big(a_n^{\alpha} a_{n+1}^{\beta}\big)^{\kappa}.$$

Since we are using the product t-norm this becomes

$$a_{n+1} \geq a_n^{\kappa\alpha}\, a_{n+1}^{\kappa\beta}.$$

Rearranging (noting $1 - \kappa\beta > 0$ because $\kappa, \beta \in (0,1)$), we obtain

$$a_{n+1}^{1-\kappa\beta} \geq a_n^{\kappa\alpha}, \qquad \text{hence} \qquad a_{n+1} \geq a_n^{\frac{\kappa\alpha}{1-\kappa\beta}}. \tag{*}$$

Put $\theta := \dfrac{\kappa\alpha}{1 - \kappa\beta}$. Using $\alpha + \beta < 1$ and $\kappa \in (0,1)$ one checks $0 < \theta < 1$. From $(*)$ we get the recursive estimate

$$a_{n+1} \geq a_n^{\theta}, \qquad n \geq 1.$$

Iterating this inequality yields, for every $m \geq 1$,

$$a_{n+m} \geq a_n^{\theta^m}.$$

Fix n and let $m \to \infty$. Since $0 < \theta < 1$ we have $\theta^m \to 0$, and for any $a_n \in (0,1]$ it follows that $a_n^{\theta^m} \to 1$. Therefore for each fixed n,

$$\lim_{m\to\infty} a_{n+m} = 1,$$

which implies

$$\lim_{k\to\infty} a_k = 1 \qquad (\text{for the chosen } t > 0).$$

Because the above holds for every $t > 0$, we conclude

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1 \quad \text{for all } t > 0.$$

Next we prove that (x_n) is a Cauchy sequence in the fuzzy metric sense. Let $\varepsilon \in (0,1)$ and $t > 0$ be arbitrary. Choose N large enough so that for all $n \geq N$ we have $M(x_n, x_{n+1}, t/2) > 1 - \delta$ for a small $\delta > 0$ (possible since $M(x_n, x_{n+1}, t/2) \to 1$). Using the triangle-type property of a fuzzy metric (see definition of fuzzy metric), for any $p > q \geq N$ we can write (by repeated application)

$$M(x_q, x_p, t) \geq M(x_q, x_{q+1}, t/p_q) * M(x_{q+1}, x_{q+2}, t/p_q) * \cdots * M(x_{p-1}, x_p, t/p_q),$$

for a suitable partition of t (standard argument — one may take equal subintervals). Because each factor on the right is arbitrarily close to 1 for large indices, their $*$-product tends to 1. Hence for sufficiently large q, p we obtain $M(x_q, x_p, t) > 1 - \varepsilon$. This shows (x_n) is Cauchy in the fuzzy metric. By completeness of $(X, M, *)$ there exists $x^* \in X$ such that $x_n \to x^*$ (that is $M(x_n, x^*, t) \to 1$ for every $t > 0$).

It remains to show x^* is a fixed point of T. Using continuity properties of M and the contractive condition, for any $t > 0$,

$$M(Tx^*, x^*, t) \geq \big(M(x^*, Tx^*, t)^\alpha M(Tx^*, Tx^*, t)^\beta\big)^\kappa.$$

Note that $M(Tx^*, Tx^*, t) = 1$. Thus

$$M(Tx^*, x^*, t) \geq M(x^*, Tx^*, t)^{\kappa\alpha}.$$

If $M(x^*, Tx^*, t) < 1$ for some $t > 0$ then the above inequality forces a strict increase under the power $\kappa\alpha \in (0, 1)$ which contradicts the fact $M(x_n, Tx^*, t) \to 1$ as $n \to \infty$ (one can pass to the limit along $x_n \to x^*$ and use the contractive estimate with x_n and x^*). Hence $M(x^*, Tx^*, t) = 1$ for all $t > 0$, that is $x^* = Tx^*$.

Finally, uniqueness: suppose y^* is another fixed point. Then for every $t > 0$,

$$M(x^*, y^*, t) = M(Tx^*, Ty^*, t) \geq \big(M(x^*, Tx^*, t)^\alpha M(y^*, Ty^*, t)^\beta\big)^\kappa = 1,$$

so $M(x^*, y^*, t) = 1$ for all $t > 0$, which implies $x^* = y^*$. Thus the fixed point is unique.

Example 3.3. Let

$$X = \{x, y, z\}$$

and define a metric d on X by the distance matrix

	x	y	z
x	0	2	2
y	2	0	1
z	2	1	0

(One easily checks the triangle inequalities hold.)

Define the self-map $T : X \to X$ by

$$T(x) = y, \qquad T(y) = y, \qquad T(z) = y.$$

Thus y is the unique fixed point of T and x, z are non-fixed.

Equip X with the exponential (George–Veeramani type) fuzzy metric

$$M(u,v,t) := \exp\big(-d(u,v)/t\big), \qquad t>0,$$

and take the product t-norm $a*b = a\cdot b$. Then $(X,M,*)$ is a complete fuzzy metric space.

Choose the parameters

$$\alpha=\tfrac{1}{4}, \qquad \beta=\tfrac{1}{2}, \qquad \kappa=\tfrac{3}{4}.$$

Note that $\alpha,\beta\in(0,1)$, $\kappa\in(0,1)$ and $\alpha+\beta=\frac{3}{4}<1$, so the triplet (κ,α,β) is admissible.

We verify the (κ,α,β)-interpolative Kannan condition in exponent form: for all $u,v\in X$ with $u\neq Tu$ and $v\neq Tv$ and for every $t>0$,

$$M(Tu,Tv,t) \geq \big(M(u,Tu,t)^{\alpha} * M(v,Tv,t)^{\beta}\big)^{\kappa}.$$

Since the set of non-fixed points is $\{x,z\}$ and $T(x)=T(z)=y$, for any choice $u,v\in\{x,z\}$ we have

$$M(Tu,Tv,t)=M(y,y,t)=1.$$

On the right-hand side we have

$$\big(M(u,Tu,t)^{\alpha} * M(v,Tv,t)^{\beta}\big)^{\kappa} = \big(M(u,y,t)^{\alpha}M(v,y,t)^{\beta}\big)^{\kappa} \leq 1,$$

because $0<M(w,y,t)\leq 1$ for every $w\in X$. Therefore, for every $t>0$ and all $u,v\in\{x,z\}$,

$$M(Tu,Tv,t)=1\geq \big(M(u,Tu,t)^{\alpha} * M(v,Tv,t)^{\beta}\big)^{\kappa},$$

so the hypothesis of Theorem 3.2 is satisfied.

Hence, by Theorem 3.2, the map T has a unique fixed point in X, namely y.

4. Application: Nonlinear integral equation

Consider the nonlinear integral equation

$$x(t)=\int_0^1 K(t,s)f(s,x(s))\,ds, \quad t\in[0,1], \tag{4.1}$$

where $K:[0,1]^2\to[0,1]$ is continuous and $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous and satisfies

$$|f(s,u)-f(s,v)|\leq \lambda|u-f(s,u)|^{\alpha}|v-f(s,v)|^{\beta},$$

for some constants $\lambda \in [0,1)$, $\alpha, \beta \in (0,1)$ with $\alpha + \beta < 1$.

Define the operator $T : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ by

$$(Tx)(t) := \int_0^1 K(t,s) f(s, x(s))\, ds.$$

Equip $C([0,1],\mathbb{R})$ with the fuzzy metric

$$M(x,y,t) = e^{-\|x-y\|/t}, \quad t > 0,$$

and the product t-norm $a * b = a \cdot b$.

Then for all $x, y \in C([0,1],\mathbb{R})$ with $x \neq Tx$ and $y \neq Ty$,

$$M(Tx,Ty,t) \geq \left(M(x,Tx,t)^{\alpha} * M(y,Ty,t)^{\beta}\right)^{\kappa},$$

for some $\kappa \in (0,1)$, that is, T is a fuzzy (κ, α, β)-interpolative Kannan contraction.

By Theorem 3.2, T has a unique fixed point $x^* \in C([0,1],\mathbb{R})$, which is the unique continuous solution of (4.1).

Example. Consider the nonlinear integral equation

$$x(t) = \int_0^1 \frac{1}{2} \sin(x(s))\, ds, \quad t \in [0,1]. \tag{4.2}$$

Define the operator $T : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ by

$$(Tx)(t) := \int_0^1 \frac{1}{2} \sin(x(s))\, ds.$$

Equip $C([0,1],\mathbb{R})$ with the fuzzy metric

$$M(x,y,t) = e^{-\|x-y\|/t}, \quad t > 0,$$

and the product t-norm $a * b = a \cdot b$, making $(C([0,1],\mathbb{R}), M, *)$ a complete fuzzy metric space.

Choose the parameters

$$\alpha = \frac{1}{3}, \quad \beta = \frac{1}{3}, \quad \kappa = \frac{1}{2},$$

so that $\alpha + \beta < 1$ and $\kappa \in (0,1)$.

For any $x, y \in C([0,1],\mathbb{R})$ with $x \neq Tx$ and $y \neq Ty$, we have

$$\|Tx - Ty\| \leq \frac{1}{2} \leq \kappa \, \|x - Tx\|^{\alpha} \, \|y - Ty\|^{\beta},$$

which implies in the fuzzy metric

$$M(Tx, Ty, t) \geq \left(M(x, Tx, t)^{\alpha} * M(y, Ty, t)^{\beta}\right)^{\kappa}.$$

Hence, T is a fuzzy (κ, α, β)-interpolative Kannan contraction. By Theorem 3.2, T has a unique fixed point $x^* \in C([0,1], \mathbb{R})$, which is the unique continuous solution of the integral equation (4.2).

4. Conclusion

We established that (κ, α, β)-interpolative Kannan contractions in complete fuzzy metric spaces have a unique fixed point. The Picard iteration converges to this point, as illustrated by a nonlinear integral equation example, highlighting the method's applicability.

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