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TRANSFORMATION FORMULAS FOR LAURICELLA'S FOURTH FUNCTION Φ_D

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Abstract: In the paper a very general transformation formula for Lauricella fourth function Φ_D of n variables has been established.

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1. Introduction, Notations and Definitions

As usual, q- series notations for complex number a and q such that $|q| < 1$ are defined as

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty} = (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad n \geq 0,$$

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r)$$

$$(a_1,a_2,\ldots,a_r;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n.$$

Also,

$$(a_1,a_2,\ldots,a_r;q)_\infty = \prod_{r=1,i=1-r}^{\infty}(a_i;q)_\infty.$$

Lauricella fourth function Φ_D is defined as,

$$\Phi_D\left[a:b_1,b_2,\ldots,b_n;c;q;z_1,z_2,\ldots,z_n\right]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a;q)_{m_1+\cdots+m_n}(b_1;q)_{m_1}(b_2;q)_{m_2}\cdots(b_n;q)_{m_n};q;z_1^{m_1}z_2^{m_2}\cdots z_n^{m_n}}{(c;q)_{m_1+\cdots+m_n}(q;q)_{m_1}(q;q)_{m_2}\cdots(q;q)_{m_n}}. \quad (1.1)$$

q-Binomial theorem is defined as,

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty} \quad (1.2)$$

[3; App. IV (IV. II)]

Transformation

In this section a transformation formula related to Φ_D has been established. Idea has been taken from [2. Theorem (1.3.1) p.6].

Let us consider

$$\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a;q^h)_{m_1+\cdots+m_n}}{(c;q^h)_{m_1+\cdots+m_n}}\frac{(b_1;q^h)_{m_1}z_1^{m_1}}{(q^h;q^h)_{m_1}}\frac{(b_2;q^h)_{m_2}z_2^{m_2}}{(q^h;q^h)_{m_2}}\cdots\frac{(b_n;q^h)_{m_n}z_n^{m_n}}{(q^h;q^h)_{m_n}}$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a;q^h)_\infty(cq^{h(m_1,\ldots,m_n)};q^h)_\infty}{(aq^{h(m_1,\ldots,m_n)};q^h)_\infty(c;q^h)_\infty}\frac{(b_1;q^h)_{m_1}z_1^{m_1}}{(q^h;q^h)_{m_1}}\cdots\frac{(b_n;q^h)_{m_n}z_n^{m_n}}{(q^h;q^h)_{m_n}}$$

$$= \frac{(a;q^h)_\infty}{(c;q^h)_\infty}\sum_{m_1,\ldots,m_n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(c/a;q^h)_r a^r q^{rh(m_1+\ldots+m_n)}}{(q^h;q^h)_r}\frac{(b_1;q^h)_{m_1}z_1^{m_1}}{(q^h;q^h)_{m_1}}\ldots\frac{(b_n;q^h)_{m_n}z_n^{m_n}}{(q^h;q^h)_{m_n}}$$

if $|aq^{h(m_1+\ldots+m_n)}|<1$, max. $(|z_1|,\ldots,|z_n|)<1$ then we have

$$= \frac{(a;q^h)_\infty}{(c;q^h)_\infty}\sum_{r=0}^{\infty}\frac{(c/a;q^h)_r a^r}{(q^h;q^h)_r}\sum_{m_1=0}^{\infty}\frac{(b_1;q^h)_{m_1}(z_1q^{rh})^{m_1}}{(q^h;q^h)_{m_1}}\ldots\sum_{m_n=0}^{\infty}\frac{(b_n;q^h)_{m_n}(z_nq^{rh})^{m_n}}{(q^h;q^h)_{m_n}}$$

$$= \frac{(a;q^h)_\infty}{(c;q^h)_\infty}\sum_{r=0}^{\infty}\frac{(c/a;q^h)_r a^r}{(q^h;q^h)_r}\frac{(b_1z_1q^{rh};q^h)_\infty}{(z_1q^{rh};q^h)_\infty}\ldots\frac{(b_nz_nq^{rh};q^h)_\infty}{(z_nq^{rh};q^h)_\infty}$$

$$= \frac{(a;q^h)_\infty (b_1z_1;q^h)_\infty ... (b_nz_n;q^h)_\infty}{(c;q^h)_\infty (z_1;q^h)_\infty ... (z_n;q^h)_\infty} \sum_{r=0}^{\infty} \frac{(c/a;q^h)_r (z_1;q^h)_r}{(q^h;q^h)_r (b_1z_1;q^h)_r} ... \frac{(z_n;q^h)_r a^r}{(b_nz_n;q^h)_r}$$

$$= \frac{(a, b_1z_1, ..., b_nz_n;q^h)_\infty}{(c, z_1, ..., z_n;q^h)_\infty} \, {}_{n+1}\Phi_n \left[\begin{array}{l} c/a, z_1, z_2, ..., z_n; q^h; a \\ b_1z_1, b_2z_2, ..., b_nz_n \end{array} \right] \tag{1.3}$$

2. Second Transformation Formula for Lauricella Function Φ_D

In this section a new transformation formula for Φ_D has been established. let us consider

$$\sum_{m_1,..,m_n=0}^{\infty} \frac{(a;q^h)_{m_1+...+m_n}}{(c;q^h)_{m_1+...+m_n}} \frac{(b_1;q)_{m_1} z_1^{m_1}}{(q;q)_{m_1}} ... \frac{(b_n;q)_{m_n} z_n^{m_n}}{(q;q)_{m_n}}$$

$$= \sum_{m_1,..,m_n=0}^{\infty} \frac{(a;q^h)_\infty}{(c;q^h)_\infty} \frac{(cq^{h(m_1+...+m_n)};q^h)_\infty}{(aq^{h(m_1+...+m_n)};q^h)_\infty} \frac{(b_1;q)_{m_1} z_1^{m_1}}{(q;q)_{m_1}} ... \frac{(b_n;q)_{m_n} z_n^{m_n}}{(q;q)_{m_n}}$$

$$= \frac{(a;q^h)_\infty}{(c;q^h)_\infty} \sum_{m_1,...,m_n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(c/a;q^h)_r a^r q^{rh(m_1+...+m_n)}}{(q^h;q^h)_r}$$
$$\times \frac{(b_1;q)_{m_1} z_1^{m_1}}{(q;q)_{m_1}} ... \frac{(b_n;q)_{m_n} z_n^{m_n}}{(q;q)_{m_n}}$$

If $|aq^{rh(m_1+...+m_n)}| < 1$, $|z_1| < 1, ..., |z_n| < 1$ then we can write,

$$= \frac{(a;q^h)_\infty}{(c;q^h)_\infty} \sum_{r=0}^{\infty} \frac{(c/a;q^h)_r a^r}{(q^h;q^h)_r} \frac{(b_1;q)_{m_1} q^{rhm_1} z_1^{m_1}}{(q;q)_{m_1}} ... \frac{(b_n;q)_{m_n} q^{rhm_n} z_n^{m_n}}{(q;q)_{m_n}}$$

Now, applying (1.2) we have

$$\frac{(a;q^h)_\infty}{(c;q^h)_\infty} \sum_{r=0}^{\infty} \frac{(c/a;q^h)_r a^r}{(q^h;q^h)_r} \frac{(b_1z_1q^{rh};q)_\infty}{(z_1q^{rh};q)_\infty} ... \frac{(b_nz_nq^{rh};q)_\infty}{(z_nq^{rh};q)_\infty}$$

$$= \frac{(a;q^h)_\infty}{(c;q^h)_\infty} \sum_{r=0}^{\infty} \frac{(c/a;q^h)_r a^r}{(q^h;q^h)_r} \frac{(b_1z_1;q)_\infty (z_1;q)_{rh}}{(b_1z_1;q)_{rh} (z_1;q)_\infty} ... \frac{(b_nz_n;q)_\infty (z_n;q)_{rh}}{(b_nz_n;q)_{rh} (z_n;q)_\infty}$$

$$= \frac{(a;q^h)_\infty (b_1z_1, b_2z_2, ..., b_nz_n;q)_\infty}{(c;q^h)_\infty (z_1;q)_\infty (z_2;q)_\infty ... (z_n;q)_\infty} \sum_{r=0}^{\infty} \frac{(c/a;q^h)_r}{(q^h;q^h)_r} \frac{(z_1;q)_{rh} ... (z_n;q)_{rh} a^r}{(b_1z_1;q)_{rh} ... (b_nz_n;q)_{rh}}$$

$$= \frac{(a;q^h)_\infty (b_1z_1, b_2z_2, ..., b_nz_n;q)_\infty}{(c;q^h)_\infty (z_1, z_2, ..., z_n;q)_\infty} \sum_{r=0}^{\infty} \frac{(c/a;q^h)_r}{(q^h;q^h)_r} \frac{(z_1, z_2, ..., z_n;q)_{rh} a^r}{(b_1z_1, b_2z_2, ..., b_nz_n;q)_{rh}} \tag{2.1}$$

3. Special cases

In this section we discuss the special cases of (1.3) and (2.1).

(a) If we put $h = 1$ in (1.3) or (2.1) both give following transformation

$$\Phi_D[a; b_1, b_2, ..., b_n; c; q; z_1, z_2, ..., z_n]$$
$$= \frac{(a, b_1z_1, b_2z_2, ..., b_nz_n; q)_\infty}{(c, z_1, z_2, ..., z_n; q)_\infty} {}_{n+1}\Phi_n \left[\begin{array}{l} c/a, z_1, z_2, ..., z_n; q; a \\ b_1z_1, b_2z_2, ..., b_nz_n \end{array} \right], \tag{3.1}$$

which is a known result [1; (1972 b) p. 621].
Again, taking $b_1z_1 = z_2, b_2z_2 = z_3, ..., b_{n-1}z_{n-1} = z_n$ in (3.1) we get,

$$\Phi_D[a; z_2/z_1, z_3/z_2, ..., z_n/z_{n-1}, b_n; c; q; z_1, z_2, ..., z_n]$$
$$= \frac{(a, b_nz_n; q)_\infty}{(c, z_1; q)_\infty} {}_2\Phi_1 \left[\begin{array}{l} c/a, z_1; q; a \\ b_nz_n \end{array} \right], \tag{3.2}$$

Taking $b_nz_n = z_1$ in (3.2) we get

$$\Phi_D[a; z_2/z_1, z_3/z_2, ..., z_n/z_{n-1}, z_1/z_n; c; q; z_1, z_2, ..., z_{n-1}, z_n]$$
$$= \frac{(a; q)_\infty}{(b; q)_\infty} \frac{(c; q)_\infty}{(a; q)_\infty} = 1. \tag{3.3}$$

Again, taking $b_nz_n = c/a$ in (3.2) we obtain,

$$\Phi_D[a; z_2/z_1, z_3/z_2, ..., z_n/z_{n-1}, b_n; ab_nz_n; q; z_1, z_2, ..., z_{n-1}, z_n]$$
$$= \frac{(a, b_nz_n; q)_\infty}{(ab_nz_n, z_1; q)_\infty} \frac{(az_1; q)_\infty}{(a; q)_\infty}$$
$$= \frac{(az_1, c/a; q)_\infty}{(z_1, c; q)_\infty}. \tag{3.4}$$

Putting $n = 2$ in (3.1) we get,

$$\Phi_D[a; b_1, b_2; c; q; z_1, z_2]$$
$$= \frac{(a, b_1z_1, b_2z_2; q)_\infty}{(c, z_1, z_2; q)_\infty} {}_3\Phi_2 \left[\begin{array}{l} c/a, z_1, z_2; q; a \\ b_1z_1, b_2z_2 \end{array} \right]. \tag{3.5}$$

Putting $b_2z_2 = c/a$ in (3.5) we have

$$\Phi_D[a; b_1, b_2; ab_2z_2; q; z_1, z_2]$$
$$= \frac{(a, b_1z_1, c/a; q)_\infty}{(c, z_1, z_2; q)_\infty} {}_2\Phi_1 \left[\begin{array}{l} z_1, z_2; q; a \\ b_1z_1 \end{array} \right]. \tag{3.6}$$

Taking $b_1/z_2 = a$ in (3.6) we find

$$\begin{aligned}\Phi_D[a; az_2, b_2; ab_2z_2; q; z_1, z_2]& \\ &= \frac{(a, b_1z_1, c/a; q)_\infty}{(c, z_1, z_2; q)_\infty}\frac{(az_2, az_1; q)_\infty}{(a, az_1z_2; q)_\infty} \\ &= \frac{(c/a, az_1, az_2; q)_\infty}{(c, z_1, z_2; q)_\infty}.\end{aligned} \tag{3.7}$$

If we take $h = 2$ in (1.3) we obtain

$$\begin{aligned}&\sum_{m_1,\dots,m_n=0}^{\infty}\frac{(a; q^2)_{m_1+\dots+m_n}}{(c; q^2)_{m_1+\dots+m_n}}\frac{(b_1; q^2)_{m_1}}{(q^2; q^2)_{m_1}}\dots\frac{(b_n; q^2)_{m_n}}{(q^2; q^2)_{m_n}}z_1^{m_1}\dots z_n^{m_n} \\ &= \frac{(a, b_1z_1, \dots, b_nz_n; q^2)_\infty}{(c, z_1, z_2, \dots, z_n; q^2)_\infty}\ {}_{n+1}\Phi_n\left[\begin{array}{l} c/a, z_1, z_2, \dots, z_n; q^2; a \\ b_1z_1, b_2z_2, \dots, b_nz_n\end{array}\right].\end{aligned} \tag{3.8}$$

For $h = 2$, (2.1) yields

$$\begin{aligned}&\sum_{m_1,\dots,m_n=0}^{\infty}\frac{(a; q^2)_{m_1+\dots+m_n}}{(c; q^2)_{m_1+\dots+m_n}}\frac{(b_1; q)_{m_1}}{(q; q)_{m_1}}\dots\frac{(b_n; q)_{m_n}}{(q; q)_{m_n}}z_1^{m_1}\dots z_n^{m_n} \\ &= \frac{(a; q^2)_\infty(b_1z_1, b_2z_2, \dots, b_nz_n; q)_\infty}{(c; q)_\infty(z_1, z_2, \dots, z_n; q)_\infty}\ {}_{2n+1}\Phi_{2n}\left[\begin{array}{l} c/a, z_1, z_1q, \dots, z_n, z_nq; q^2; a \\ b_1z_1, b_1z_1q, \dots, b_nz_n, b_nz_nq\end{array}\right].\end{aligned} \tag{3.9}$$

For $n = 2$, (3.9) yields

$$\begin{aligned}&\sum_{m_1,m_2=0}^{\infty}\frac{(a; q^2)_{m_1+m_2}}{(c; q^2)_{m_1+m_2}}\frac{(b_1; q)_{m_1}}{(q; q)_{m_1}}\frac{(b_2; q)_{m_2}}{(q; q)_{m_2}}z_1^{m_1}z_2^{m_2} \\ &= \frac{(a; q^2)_\infty(b_1z_1, b_2z_2; q)_\infty}{(c; q^2)_\infty(z_1, z_2; q)_\infty}\ {}_5\Phi_4\left[\begin{array}{l} c/a, z_1, z_1q, z_2, z_2q; q^2; a \\ b_1z_1, b_1z_1q, b_2z_2, b_2z_2q\end{array}\right].\end{aligned} \tag{3.10}$$

Taking $b_2 = 1$, $b_1 = b$ and $z_1 = z$ in (3.10) we get

$$\begin{aligned}&{}_3\Phi_2\left[\begin{array}{l} \sqrt{a}, -\sqrt{a}, b; q; z \\ \sqrt{c}, -\sqrt{c}\end{array}\right] \\ &= \frac{(a; q^2)_\infty(bz; q)_\infty}{(c; q^2)_\infty(z; q)_\infty}\ {}_3\Phi_2\left[\begin{array}{l} c/a, z, zq; q^2; a \\ bz, bzq\end{array}\right],\end{aligned} \tag{3.11}$$

This is a transformation of a ${}_3\Phi_2$ series on base q into another ${}_3\Phi_2$ series on base q^2

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